SUBSCRIPTION
$2.00 PER YEAR
IN ADVANCE
SINGLE COPIES
25c.

All Business
Communications
should be addressed
to the
Editor and Manager

VOL. XV UNIVERSITY, LA., DECEMBER, 1940. No. 3

Entered as second-class matter at University, Louisiana.

Published monthly excepting June, July, August, September, by LOUISIANA STATE UNIVERSITY, Vols. 1-8 Published as MATHEMATICS NEWS LETTER.

THIS JOURNAL IS DEDICATED TO THE FOLLOWING AIMS: (1) Through published standard papers on the culture aspects, humanism and history of mathematics to deepen and to widen public interest in its values. (2) To supply an additional medium for the publication of expository mathematical articles. (3) To promote more scientific methods of teaching mathematics. (4) To publish and to distribute to groups most interested high-class papers of research quality representing all mathematical fields.

Every paper on technical mathematics offered for publication should be submitted (with enough enclosed postage to cover two two-way transmissions) to the Chairman of the appropriate Committee, or to a Committee member whom the Chairman may designate to examine it, after being requested to do so by the writer. If approved for publication, the Committee will forward it to the Editor and Manager at Baton Rouge, who will notify the writer of its acceptance for publication. If the paper is not approved the Committee will so notify the Editor and Manager, who will inform the writer accordingly.

1 All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

2. The name of the Chairman of each committee is the first in the list of the committee.

3. All manuscripts should be worded exactly as the author wishes them to appear in the MAGAZINE.

Papers intended for the Teacher's Department, Department of History of Mathematics, Bibliography and Reviews, or Problem Department should be sent to the respective Chairmen.

Committee on Algebra and Number Theory: L. E. Bush, W. Vann Parker, R. F. Rinehart.

Committee on Analysis and Geometry: W. E. Byrne, Wilson L. Miser, Dorothy McCoy, H. L. Smith, V. Thébault.

Committee on Teaching of Mathematics: Joseph Seidlin, James McGiffert.

Committee on Statistics: C. D. Smith, Irby C. Nichols.

Committee on Mathematical World News: L. J. Adams.

Committee on Bibliography and Reviews: H. A. Simmons.

Committee on Problem Department: R. C. Yates, E. P Starke.

Committee on Humanism and History of Mathematics: G. Waldo Dunnington.

PUBLISHED BY THE LOUISIANA STATE UNIVERSITY PRESS

A FINAL APPEAL

C. D. Smith, Chairman of the Louisiana-Mississippi Section of M. A. of A. has issued a strong plea to mathematicians of these two states to be at the Baton Rouge joint meetings now less than 30 days away. In the closing paragraph, he says:

> "Did you ever know a mathematician who refused to lend interest and aid to his country?.... Our government must have our aid now and we are going to Baton Rouge to find out what American Mathematicians can do for America. Our boys in the Army and our government officials know that we are ready...."

Every American mathematical worker should be sensitive to such an appeal. We of the Southland should react to it with special concern. A mere handful of Mathematicians trickling into Baton Rouge from Louisiana, Mississippi, Arkansas, Alabama, Tennessee, Texas, or other nearby states will be a far cry from what is desired. A huge army of the mathematics-minded should pour into this centre of the Deep South, eager—eager, even in the face of financial handicaps—to learn, and to serve our common cause, a cause which once more becomes invested with near-patriotic elements.

Come to these meetings at University, Louisiana!* We *sacrifice* in order to gain many other good things. Is not the series of prepared programs in Mathematics worth every possible sacrifice if by it we can reap through our presence at them a harvest of inspiration and knowledge in our life's profession?

S. T. SANDERS.

*Dormitory reservations may be made at any time prior to the meetings.

New Criteria for Accuracy in Approximating Real Roots by the Newton-Raphson Method

By Myron G. Pawley
Colorado School of Mines

ABSTRACT

The expression commonly given in the literature for the inherent error involved in the Newton-Raphson method is incorrect. New criteria are here derived which may be safely used, and which may be applied likewise to extensions of the Newton-Raphson method involving derivatives of higher order.

Newton's method, as it is generally referred to in the literature, is really not Newton's method at all, but a modification of it developed by Joseph Raphson.* Florian Cajori gives the honor of invention to Francis Vieta (1540-1603) who developed a method closely resembling Newton's, but he suggests calling the process the "Newton-Raphson Method" since this designation more nearly represents the facts of history.

During the seventeenth century and later mathematicians were concerned over the insecurity in the Newton-Raphson process since successive corrections did not always yield results converging to the true value of the root sought. Lagrange pronounced the method insecure and believed the *a priori* determination of the conditions under which the method can be used safely to be difficult if not impossible. It is interesting to note that both Newton and Lagrange approached the problem of approximation by purely analytical considerations. It was not until J. R. Mourraille† and later Joseph Fourier‡ introduced geometrical considerations that analytical criteria were developed which are sufficient to insure security in the operation of the Newton-Raphson method.

Mathematicians have also been concerned with the fact that the method does not reveal, without the application of special tests, how many digits in the approximation are corrrect. Many special tests

*Cajori, *A History of the Arithmetical Methods of Approximation to the Roots of Numerical Equations of one Unknown Quantity.* Colorado College Publication, General Series Nos. 51, 52, 1910. The historical references in this paper are taken largely from this publication which includes many references on the subject.

†Cajori, *op. cit.*, p. 207.

‡Cajori, *op. cit.*, p. 208.

have been advanced.* One need only glance at these to realize how cumbersome and impractical they are. One seemingly "practical" test was given by Maseres† in 1807 to the effect that if the answer is correct to n places, the next step in the Newton-Raphson method is correct to $2n$ places. This loosely stated criterion has been given by many and appears in more elaborate form in recent texts.‡ One purpose in writing this paper is to point out that, as given, this criterion for accuracy in the use of the Newton-Raphson method is incorrect. Another is to develop criteria which may be safely used, and which may be applied likewise to extensions of the Newton-Raphson method involving derivatives of higher order. The particular analytical approach which has been so extensively employed cannot be used in developing criteria for accuracy in these cases. Perhaps this is the reason why no criteria have been developed for these extended methods.

Workers in this field apparently have not followed the precedent established by Mourraille and Fourier who contributed so much by their geometrical approach to the problem. They have used, instead, the purely analytical attack which had led Lagrange to conclude that the determination of the conditions under which the method can be used safely was difficult if not impossible. Cajori repeatedly remarks that the all-important question of the convergence of series expansions involved was not amply considered by the mathematicians of the 18th century. It is precisely this oversight which has led to the error cited above.

In the derivation of an expression for the inherent error in the Newton-Raphson process the general procedure§ has involved a series expansion for the error $h-h_1$ where h is the true correction to be applied to a, and h_1 is the Newton-Raphson correction. Two terms of this series gives the

$$\text{(1)} \qquad \text{Error} = h - h_1 = -\frac{Mh_1^2}{2f'(a)} + \frac{M^2h_1^3}{2[f'(a)]^2} + \cdots$$

*E. Schröder, *Math. Annalen*, Vol. 2, 1870, p. 317
E. Netto, *Math. Annalen*, Vol. 29, 1887, p. 141.
C. Isenkrahe, *Math. Annalen*, Vol. 31, 1888, p. 309.
F. Franklin, *Am. Jour. of Math.*, Vol. 4, 1881, p. 275.

†Cajori, *op. cit.*, p. 205.

‡L. E. Dickson, *First Course in the Theory of Equations*, John Wiley and Sons, New York, 1922, p. 95.
FR. A. Willers, *Methoden der Praktischen Analysis*, Walter de Gruyter and Co., Berlin, 1928, p. 172.
J. C. Scarborough, *Numerical Mathematical Analysis*, Johns Hopkins Press, Baltimore, p. 182.

§J. B. Scarborough, *op. cit.*, p. 182.
FR. A. Willers, *op. cit.*, p. 172.

Terms beyond the first are summarily dismissed, "since h_1 is always a small decimal," and the error written

$$\epsilon_1 \leqq \left| \frac{Mh_1^2}{2f'(a)} \right| \tag{2}$$

where M denotes the maximum numerical value of $f''(x)$ in the neighborhood of $a+h_1$.

From (2) it follows that if

$$\left| \frac{M}{2f'(a)} \right| \leqq 1$$

the error $\epsilon_1 \leqq h_1^2$ and we have the rule often given:* RULE. If h_1 when expressed as a decimal, has k zeros between the decimal point and the first significant figure, the division may be safely carried to $2k$ decimal places.

Now inspection of (1) shows that if the ratio of M to $f'(a)$ is large, even though h_1 be small, the second term may be far from negligible. In fact closer examination of the series involved shows that if

$$\frac{2Mh_1}{f'(a)}$$

is numerically greater than 1 the series will not even converge! We shall see later that even if

$$\left| \frac{M}{2f'(a)} \right| \leqq 1\dagger$$

this rule may not be safe.

The following is an example for which the above expression (2) for the error ϵ_1 gives a value far too small.

Let it be required to find the approximate root of

$$x^3+297x^2-595x+296.98=0$$

between $b=1$ and $a=1.1$.

By synthetic division we find:

$$b=1 \qquad\qquad a=1.1$$

$$f(b)=-0.02 \qquad\qquad f(a)=3.181$$

$$f'(b)=2 \qquad\qquad f'(a)=62.03$$

$$\frac{f''(b)}{2}=300 \qquad\qquad \frac{f''(a)}{2}=300.3=\frac{M}{2}.$$

*L. E. Dickson, *loc. cit.*, p. 95.
Cajori, *loc. cit.*, p. 205.
†J. B. Scarborough, *loc. cit.*, p. 182.

Applying the Newton-Raphson correction to the end of the interval (a,b) where $f(x)$ and $f''(x)$ have like sign, as recommended,* we find

$$h_1 = -\frac{f(a)}{f'(a)} = -0.051.$$

The first approximation is therefore

$$a_1 = a - \frac{f(a)}{f'(a)} = 1.049$$

and the error as given by expression (2) above

$$\epsilon_1 < \left| \frac{Mh_1^2}{2f'(a)} \right|, \text{ or}$$

$$\epsilon_1 < 0.013.$$

Now the root of the above equation is actually between 1.005 and 1.006 so we see that in this case the actual error in the first Newton-Raphson approximation is more than *three times* the error given by the above expression. If we apply the "rule of thumb" as is often given in regard to the number of digits to which the Newton-Raphson correction may safely be carried† to this problem we shall likewise be in error.

We shall now develop an expression for the inherent error in the Newton-Raphson process which is always safe to apply, and one which may be likewise applied to extensions of the Newton-Raphson method involving derivatives of higher order.

We shall refer to the interval (a,b), but it should be understood that the developments to follow also hold if $b>a$.

Assuming that $f(x)$ may be expanded by Taylor's series in the interval (a,b) we may write

$$(3) \qquad f(x) = f(a+h) = f(a) + f'(a)h + \frac{f''(a)h^2}{2} + \cdots\cdots + \frac{f^{\nu}(\varepsilon)h^{\nu}}{(\nu-1)!}$$

$$\varepsilon = a + \theta h, \quad 0<\theta<1.$$

Let

$$(4) \qquad S_\nu = f(a) + f'(a)h + \cdots \frac{f^{\nu-1}(a)h^{\nu-1}}{(\nu-1)!}.$$

*Cajori, *op. cit.*, p. 207.
L. E. Dickson, *op. cit.*, p. 93.
†Cajori, *op. cit.*, p. 205.
L. E. Dickson, *op. cit.*, p. 95.

This function, which consists of the sum of the first ν terms of (3), represents a curve passing through $[a, f(a)]$ and having the same first $\nu-1$ derivatives at this point as $f(x)$. It is a curve having contact of order $\nu-1$ with $f(x)$ at $[a, f(a)]$, an x intercept of which may be taken as an approximation to the desired root of $f(x)=0$. Letting $h_{\nu-1}$ be the root of $S_\nu=0$ corresponding to the x intercept of this approximation curve nearest the desired root, and substituting in (3) we get

$$(5) \qquad f(a+h_{\nu-1}) = \frac{f^{\nu}(\varepsilon)h^{\nu}_{\nu-1}}{\nu!}, \quad \varepsilon = a+\theta h_{\nu-1}, \quad 0<\theta<1.$$

This is the ordinate of $f(x)$ at $x=a+h_{\nu-1}$, where the approximation curve S_ν crosses the x axis.

If we designate by M the maximum numerical value of $f^{\nu}(x)$ in the interval (a,b) we may write

$$(6) \qquad |f(a+h_{\nu-1})| \leqq \left|\frac{Mh^{\nu}_{\nu-1}}{\nu!}\right|, \quad |h_{\nu-1}| < |b-a|.$$

We therefore know that an upper bound to the numerical value of the ordinate of $f(x)$ at the x intercept of the approximation curve S_ν will be

$$\left|\frac{Mh^{\nu}_{\nu-1}}{\nu!}\right|$$

Dividing (6) by μ, the minimum numerical value of $f'(x)$ in the interval (a,b), we obtain an upper bound to the error involved in using $a+h_{\nu-1}$ as an approximation to the root of $f(x)=0$ in the interval (a,b):

$$(7) \qquad \epsilon \leqq \left|\frac{Mh^{\nu}_{\nu-1}}{\mu\nu!}\right| .$$

Fig. 1 illustrates the significance of this expression for the error when $\nu=2$, and when $f(x)=0$ has a single real root in the interval (a,b), and $f'(x)=0$ and $f''(x)=0$ have no real roots in the interval (a,b).

The use of $a+h_{\nu-1}$ as an approximation to the root of $f(x)=0$ and the expression (7) for the inherent error so involved is not practicable when ν is greater than 3. In this paper we shall consider only the case where $\nu=2$, deferring the detailed discussion for the case with $\nu=3$ until a later paper.

When $\nu=2$ the approximation curve S_2 becomes the tangent at $[a, f(a)]$, $a+h_1$ becomes the Newton-Raphson approximation to the

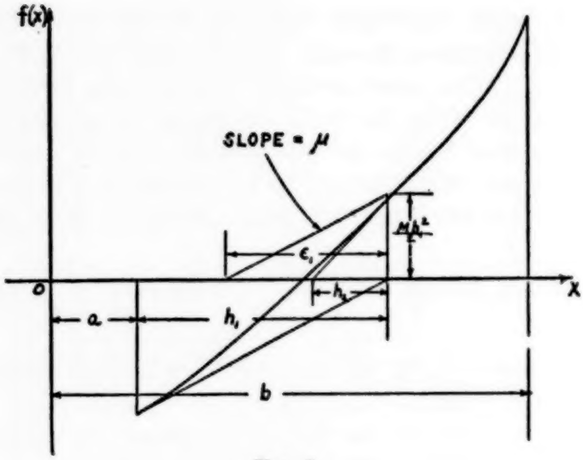

FIG. 1.

Diagram illustrating the Geometrical Significance of the Expression for the Inherent Error in the First Newton-Raphson Approximation.

desired root, and (7) then gives an upper bound to the error involved in this approximation, namely

$$\epsilon_1 < \left| \frac{Mh_1^2}{2\mu} \right|, \quad \text{(see Fig. 1).} \tag{8}$$

We have not placed restrictions upon $f(x)$ outside of the interval (a,b), but if we require that $|h_1| < |b-a|$ examination of Fig. 1 will show that each succeeding Newton-Raphson approximation will be nearer to the desired root than the preceding.

We shall now derive an expression for the approximate error involved after n successive applications of the Newton-Raphson method.

Let $h_1, h_2, \cdots h_n$ denote the successive Newton-Raphson corrections to be added to a, and $a_1, a_2, \cdots a_n$ the successive Newton-Raphson approximations to the desired root. From (8) we obtain for an upper bound to the error in approximating the root by $a_1 = a + h_1$

$$\epsilon_1 < \left| \frac{Mh_1^2}{2\mu} \right| . \tag{9}$$

Similarly, for the error involved in the nth successive approximation, $a_n = a_{n-1} + h_n$, we obtain

$$\epsilon_n < \left| \frac{Mh_n^2}{2\mu} \right|, \quad n > 1. \tag{10}$$

Since $|h_2| < \epsilon_1$, as can be seen from Fig. 1, we have from (9) and (10)

$$\epsilon_2 < \left| \left(\frac{M}{2\mu} \right)^3 \right| h_1^4$$

Furthermore, since $|h_3| < \epsilon_2$, we have

$$\epsilon_3 < \left| \frac{Mh_3^2}{2\mu} \right|, \quad \text{or}$$

$$\epsilon_3 < \left| \left(\frac{M}{2\mu} \right)^7 \right| h_1^8.$$

In fact, since $|h_n| < \epsilon_{n-1}$ we may write

$$\epsilon_n < \left| h_1^{2^n} \left(\frac{M}{2\mu} \right)^{2^n - 1} \right|. \tag{11}$$

We may now write in summary the following: THEOREM. *If $f(x) = 0$ has a single real root in the interval (a,b), and if $f'(x) = 0$, $f''(x) = 0$, and $f'''(x) = 0$ have no real roots in the interval (a,b), and if we designate by* a *that one of the numbers a and b to which successive Newton-Raphson corrections are applied, and if, furthermore, with the definitions below, $|h_1| < |b-a|$, the nth Newton-Raphson approximation to the root will be equal to a_n with an error numerically less than ϵ_n.*

$$h_1 = \frac{-f(a)}{f'(a)}.$$

a_n = *the nth Newton-Raphson approximation to the root.*

M = *Maximum numerical value $f''(x)$ in (a,b).*

μ = *Minimum numerical value of $f'(x)$ in (a,b).*

$$\epsilon_n < \left| h_1^{2^n} \left(\frac{M}{2\mu} \right)^{2^n - 1} \right|$$

The condition that $f'''(x) = 0$ have no real root in the interval (a,b), while not necessary, is sufficient to insure that M, the maximum numerical value of $f''(x)$ in the interval (a,b), will be either $f''(a)$ or $f''(b)$ whichever has the greater numerical value. This is a matter of convenience. Because no restriction is placed upon $f(x)$ outside of the interval (a,b) it is necessary to impose the condition that $|h_1| < |b-a|$ in order to insure that the succeeding approximations will converge on the desired root.

Examination of the expression for ϵ_n shows that if

$$\left|\frac{M}{2\mu}\right| \leqq 1 \quad \text{the error}$$

$$\epsilon_n < |h_1^{2^n}| \,. \tag{12}$$

This result is important because it permits us to write the following rule: RULE. *If $f(x)=0$ has a single real root in the interval (a,b), if $f'(x)=0$, $f''(x)=0$, and $f'''(x)=0$ have no real roots in the interval (a,b), and if we designate by* a *that one of the numbers a and b to which successive Newton-Raphson corrections are applied, and if, furthermore, with the definitions below,*

$$\left|\frac{M}{2\mu}\right| \leqq 1$$

and h_1 when expressed as a decimal has k zeros between the decimal point and the first significant figure, this first significant figure being less than 7, the nth Newton-Raphson correction may safely be computed to $2^n k$ decimal places.

$$h_1 = -\frac{f(a)}{f'(a)}\,, \quad |h_1| < |b-a|.$$

M = Maximum numerical value of $f''(x)$ in (a,b).

μ = Minimum numerical value of $f'(x)$ in (a,b).

This rule follows from the expression (12) for the error ϵ_n involved in the nth Newton-Raphson approximation. Each successive n placed in the expression (12) squares the preceding error involved, and thereby doubles the number of decimal places to which the approximation may safely be carried at that stage. Therefore, with k as defined in the rule, the nth approximation may safely be carried to $2^n k$ decimal places. The requirement that the first significant figure of h_1 be less than 7 merely provides that the first significant figure in the nth approximation will be less than 5 in the $2^n k+1$ decimal place.

This rule should be contrasted with one sometimes given which is incorrect* since it requires that

$$\left|\frac{M}{2f'(a)}\right| \leqq 1.$$

*J. B. Scarborough, *op. cit.*, p. 182.

This may be insufficient. The correct expression (11) for the error involved reduces to $\epsilon_n < |h_1^{2^n}|$ *only* if

$$\left|\frac{M}{2\mu}\right| \leqq 1$$

and permits us to write the above rule subject to this condition. Since $f'(a)$ may be greater numerically than μ it should not be used in the rule in place of μ.

The following problem will show how the new criteria for accuracy derived in this paper can be used to advantage in finding an approximate root.

Let it be required to find the root of $x^3 - x - 9 = 0$ between $a = 2.2$ and $b = 2.3$. By synthetic division we find

$$\begin{array}{ll} a = 2.2 & b = 2.3 \\ f(a) = -0.552 & f(b) = 0.867 \\ f'(a) = 13.52 = \mu & f'(b) = 14.87 \\ \dfrac{f''(a)}{2} = 6.6 & \dfrac{f''(b)}{2} = \dfrac{M}{2} = 6.9 \end{array}$$

The first Newton-Raphson correction to be applied to a is

$$h_1 = -\frac{f(a)}{f'(a)} = 0.0408$$

giving
$$a_1 = a - \frac{f(a)}{f'(a)} = 2.2408$$

with error
$$\epsilon_1 < \left|\frac{Mh_1^2}{2\mu}\right|, \quad \epsilon_1 < 0.000851.$$

We also know from (11) an upper bound to the error in the nth successive Newton-Raphson approximation. For example the third approximation will be in error $\epsilon_3 < 7x10^{-14}$.

Since in this problem

$$\left|\frac{M}{2\mu}\right| < 1$$

we may use the simple expression (12) for an upper bound to the error in the nth approximation, or we may apply the rule in regard to the number of decimal places to which our approximation may safely be

carried. In this example k is equal to unity and the third Newton-Raphson approximation may safely be computed to $2^n k$ or 2^3 decimal places. It should be observed that his simple rule, based on the expression (12), gives an upper bound to the error which, although always safe, may be considerably larger than the upper bound determined from the somewhat more complicated expression (11). The latter expression, giving an upper bound to the error in the nth Newton-Raphson correction, indicates that in this problem the computation of the third approximation may safely be carried to twelve decimal places.

With three successive applications of the Newton-Raphson method to the above equation the root is found to be 2.240040987469, accurate to the twelfth decimal place as indicated by the test for accuracy developed in this paper.

Mathematics in General Education, a publication of D. Appleton-Century Co., is a report of the committee on the function of mathematics in general education, for the Commission on Secondary School Curriculum of the Progressive Education Association. This committee is composed of the following: Albert A. Bennett, Cuthbert Daniel, Harold Fawcett, Maurice L. Hartung, Robert J. Havighurst, Joseph Jablonower, Ruth Kotinsky, and V. T. Thayer.

Fascicule XCVII of *Mémorial des Sciences Mathématiques* is entitled *Polynomes et fonctions de Legendre* and is the work of M. Rene Lagrange, who is Professor a la Faculte des Sciences de Dijon.

Transactions of the American Mathematical Society is edited by Professors William C. Graustein, Einar Hille and C. C. MacDuffee.

"Problem-solvers" would do well to become familiar with the problem department of *Boletin Matematico*, published monthly at Avenida de Mayo 560 in Buenos Aires, Republica Argentina.

Dr. A. B. Mewborn of the California Institute of Technology has been appointed assistant professor at the University of Arizona.

Professor A. E. H. Love, professor of natural philosophy in the University of Oxford (England), died on June 5, 1940 at the age of seventy-seven. Professor Love will be remembered chiefly for his work in connection with certain wave-trains and for his treatise on the *Mathematical Theory of Elasticity*.

Sir J. J. Thomson, O. M., F. R. S., died August 30, 1940. At the time of his death he was Master of Trinity College, Cambridge (England) and professor of physics in the University. He was laid to rest in Westminster Abbey.

—Reported by L. J. Adams.

Humanism and History of Mathematics

Edited by
G. Waldo Dunnington

A History of American Mathematical Journals

By Benjamin F. Finkel
Drury College

(Continued from November, 1940, issue)

The history of *The Mathematical Companion* was begun in the last issue of the Magazine. The November installment was concluded with a list of "36 questions, the 36th being a Prize Question."

Pages 6-8 contain the following:

APPENDIX:

ARITHMETICAL QUESTIONS

For the Benefit of

City and County School Masters

This collection consists of twenty-seven questions and are not numbered in the original pamphlet. Apparently, they were all proposed by the editor, John D. Williams.

1. A gentleman has a garden in the form of an equilateral triangle, the sides whereof are each 50 ft.: at each corner of the garden stands a tower; the height of A is 30 feet, that of B 34 feet, and that of C 28 feet. At what distance from the bottom of each of these towers must a ladder be placed that it may just reach the top of each tower, and what will be the length of the ladder, the ground of the garden being horizontal?

2. A says to B, give me \$100 and I shall have as much as you. No, says B to A, give me \$100 and I shall have twice as much as you. How much has each?

3. Here are two pillars in a straight line, perpendicular to the plane of the horizon, whose distance asunder is 180 feet, the one is 60

and the other 40 feet high: Query, in what part of the line of distance a ladder may be fixed, so as to reach the top of each pillar without removing it at the bottom, also the length of the ladder.

4. One being asked the hour of the day, answered that the time passed from noon was equal to $^2/_{13}$ of the time remaining from midnight: I demand what o'clock.

5. A person had two silver cups of unequal weight, having one cover to both of 5 ounces; now if the cover is put on the less cup it will double the weight of the greater cup and set on the greater, it will be thrice as heavy as the less cup; what is the weight of each cup, independently of algebra or double position? (By J. D. W.)

6. Two trees standing on a horizontal plane are 120 feet asunder; the height of the higher of which is 100 feet and that of the shorter, 80; whereabouts in the plane must a person place himself so that his distance from the top of each tree and the distance of the tops themselves shall be all equal to each other?

7. A man and his wife can drink out a cask of beer in 12 days, but when the man was from home it lasted the woman 30 days, how many days would the man be in drinking it alone?

8. A and B together can perform a piece of work in 8 days, A and C together in 9 days, and B and C in 10 days; how many days will it take each person to perform the same work alone?

9. After a certain number of men had been employed on a piece of work for 24 days and had half finished it, 16 more were set on and the remaining half was completed in 16 days; how many men were employed at first and what was the whole expense at 1s. 6d. per day for each man?

10. From each of 16 pieces of gold a person filed the worth of half a crown and then offered them in payment of the original value, and the fraud being detected and the pieces weighed, they were found to be worth in whole, no more than 8 guineas; what was the original value of each piece?

11. At £19 $11^3/_4$s. per ton, what will 19 tons 19 cwt. 3 qu. $27^1/_2$ lb. come to?

12. Two companions have got a Parcel of Guineas; says A to B if you will give me one of your Guineas, I shall have as many as you have left. Nay, replied B, if you give me one of your Guineas, I shall have twice as many as you will have left. How many Guineas had each of them?

13. Two men had a mind to purchase a house rated at $200; says A to B, if you give me $^2/_3$ of your money, I shall be able to pur-

chase the house alone. But says B to A, if you give $^3/_4$ of yours, I shall be able to purchase the house alone. How much money had each of them?

14. A person being asked how old he was, answered, if I quadruple $^2/_3$ of my years, and add one half of them plus 50 to the product, the sum will be so much above 100 as the number of my years is now below 100. How old was he?

15. Y. Z. made the following bet for 1000 guineas, to be decided the Monday, Tuesday, and Wednesday in Whitsem-week, on Barnhan Downs, between the hours of eight in the morning, and eight at night. The proposed has 10 choice cucketters in full exercise, who on this occasion, are to be distinguished by the first 10 letters of the Alphabet. These are to run and gather up and carry singly 1000 eggs, laid in a straight line, just two yards asunder, putting them gently into a basket placed just a fathom behind the first. They are to work one at a time in the following order: A is to fetch up the first ten eggs, B the second, C the third ten, and 20 on forward to K, whose turn it will be to fetch up the 100th egg. After which A sets out again for the next 10, B takes the next, and so forward alternately, till K shall have carried up the 1000th egg, at 100 eggs per man. . . .

16. The fellows are to have £300 for their three days work, if they do it, and it is to be distributed in proportion to the ground each man shall in his course have gone over; required, first, how many miles each person will have run? Secondly, what part of the £300 will come to his share? Thirdly, whether if the men had been posted at proper places, they had not better had run from London to York twice and back in the time, taking the measure at 180 miles?

17. A lad having got 4000 nuts, in his return home was met by mad Tom, who took from him $^5/_8$ of $^2/_3$ of his whole stock. Raving Ned lights on him afterward and forced $^2/_5$ of $^5/_8$ of the remainder from him, unluckily positive Jack found him, required $^7/_{10}$ of $^{17}/_{20}$ of what he had left. Smiling Dolly was, by promise to have $^3/_4$ of $^1/_4$ of what nuts he brought home; how many then had the boy left?

18. In distress at sea, they threw out 17 hhds. of sugar worth £34 per hhd. The worth of which came up to but $^4/_7$ of the Indigo they cast overboard; besides which, they threw out 13 iron guns worth £18 10s. a piece; the value of all amounted to two-sevenths of nine-thirteenths of that and the ship and loading; what value came into the port?

19. A person dying, left his wife with child, and making his will, ordered, that if she went with a son, $^2/_3$ of the estate should belong to

him, and the remainder to his mother; and if she went with a daughter, he appointed the mother $^2/_3$ and the girl $^1/_3$: But it happened that she was delivered both of a son and daughter by which she lost in equity £2000 more than if there had been only a girl: what would have been her dowery had she had only a son?

20. There are two columns of the ruins of Per Sepolis, left standing upright, one is 64 feet above the plane, the other 50. Between these, in a right line, stands an ancient statue, the head whereof is 97 feet from the summit of the higher, and 86 feet from the top of the lower column; the base whereof measures just 76 feet to the center of the figure's base. Required the distances of the tops of the columns.

21. I would plant 10 acres of hop ground which must be done either in the square order, as the number 4 stands on the dice, or in the Quincunx order, as the number 5; the three nearest bulbs, in both cases, must be set lineally just 6 feet asunder: how many plants more will be required for the last order than for the first, admitting the form of the plot to lay the most advantageous for the plantation in either case.

22. I have an orchard in the form of a quadranglar trapezoid, containing $3^3/_4$ acres, which being divided by a diagonal, or line from corner to corner, the perpendicular of one of the triangles is 430 links and the other 360. The length of the said diagonal, or common base of those triangles is required?

23. In turning a one horse chaise within a ring of a certain diameter, it was observed that the outer wheel made two turns, while the inner made but one; the wheels were equally high, and supposing them fixed at the suitable distance, or 5 feet asunder on the axle-tree; pray what was the circumference of the track discribed by the outer wheel?

24. The Moon is a globe in diameter 2170 miles: I required how many quartes of wheat she would contain if hollow, 2150 solid inches being the bushel; and how much yard-wide stuff make her a waistcoat was she to be clothed?

25. Hiero, King of Sicily, ordered his Jeweller to make him a crown containing 63 ounces of gold; the workmen thought of substituting part silver therein, to have a proper prerequisite, which taking air, Archimedes was appointed to examine it, who, putting it into a vessel of water, found it raised the fluid, or that itself contained 82,245 cubic inches of metal; and having discovered that the cubic inch of gold, more cubically, weighed 10.36 ounces, and that of silver but 5. 85 ounces; he, by calculation, found what part of his Majesty's gold had been changed.

26. If 9 gentlemen or 15 ladies will eat 17 apples in 5 hours, and 15 gentlemen and 9 ladies can eat up 47 apples of a similar size in 12 hours, the apples growing uniformly; how many boys will eat of 360 apples in 60 hours, admitting that 120 boys can eat the same number as 18 gentlemen and 26 ladies. (By J. D. Williams.)

27. Divide 21 into two such parts, so that if the greater be divided by the less and the less by the greater, and the greater quotient multiplied by 5, and the less by 125, the products shall be equal without supposing 14 and 7, and 15 and 6.

This pamphlet seems to be really the first number of Williams' *Mathematical Companion*, although it is stated on the inside of the front cover that "No. 1 will be published on the first of May next." (1829).

Bolton gives Harvard University Library as the only Library which contains a copy of this journal and he gives the date of its publication, 1829-1831. All the information we have concerning this magazine, we obtained from the copy in the Library of Harvard University. This copy was loaned to the writer by Professor W. E. Byerly. In a letter to the writer by Professor Byerly, June 16, 1909, Professor Byerly says, "I have had the Harvard Library searched high and low for the *Mathematical Companion* with the result of unearthing among our stray pamphlets the enclosed copy apparently a prospectus of the proposed publication. It is the only number of the *Companion* that we possess."

The editor, John D. Williams, in addition to editing several elementary mathematical text books, such, for example, as J. R. Young's *Elements of Analytical Geometry*, (Philadelphia, 1833) wrote *An Elementary Treatise on Algebra*, Boston, 1840. He was also author of a *Key to Hutton's Mathematics*.

His most important work, however, was *An Elementary Treatise on Algebra*. It is an 8 vo. and contains 605 pages. A great deal of space is given to Infinite Series and Diophantine Analysis. The last part of the book contains 670, 668 having been repeated, miscellaneous problems some of which are quite difficult,—too difficult to be put into a book intended for elementary school use. Copies of this book are very scarce. A copy of it is in the private Library of Dr. Artemas Martin, Washington, D. C. The writer also owns a copy.

In June, 1832, John D. Williams published fourteen challenge problems. Asher B. Evans, in reply to a query in *Educational Notes and Queries*,* pp. 9-10, Vol. 2, edited by W. D. Henkle, sent the editor a newspaper clipping containing Williams' 14 problems together with

his challenge to the Mathematicians of the country. It reads as follows:

Messers Editors.—It is this day six months since under the signature of *Diophantus*, I proposed through the medium of your paper, to the mathematicians of America a collection of problems in Diophantine Analysis. No correct solutions having as yet been received to the whole of them, I take this opportunity to fulfil my pledge to furnish such, and enclosed they will come to your hand. I now desire to re-propose them for the ensuing six months and shall except from my challenge the Hon. Nathaniel Bowditch, LL.D., &c., &c., of Boston, Mass., Mr. Eugene Nulty, of Philadelphia, and Professor Theodore Strong, of Rutgers College, New Brunswick, N. J., only. The list of gentlemen challenged stands then, as follows:

Professor Robert Adrain, University of Pennsylvania; Henry J. Anderson, Columbia College, New York; Benj. Peirce, Harvard University, Cambridge, Massachusetts; Mr. J. Ingorsall Bowditch, Boston, Mass.; Mr. Marcus Cotlin, Hamilton College, Clinton, New York; Mr. M. Floy, Jun., New York; Mr. C. Gill, Sawpitts Academy, New York; Mr. L. L. Inconnew, Cincinnati, Ohio; Mr. Benjamin Hollowell, Alexandra, D. C.; Mr. Charles Farquhar, Alexandra, Maryland; Mr. Samuel Ward,* 3rd, New York.

It being presumed that there are none in the United States, with the exception of the above list would think of attempting their resolution, the questions proposed are as follows:

1. Make $x^2+y^2=a^2=z^2+w^2=\square$ and $x^2-w^2=z^2-y^2=\square$.
   Ans. if $a=7585$ then $x=7400$, $y=1665$, $z=6273$, $w=4264$.
2. Make $x^2+y^2=b^2=z^2+w^2$, and $x^2-z^2=w^2-y^2=\square$.
   Ans. If $b=697$ then $x=680$, $y=153$, $z=672$, and $w=185$.
3. Make $x^2\pm 23253x=\square$, $x^2\pm 7500x=\square$ and $x^2\pm 10324x=\square$.
   Ans. $x=105625$.
4. Make $x^2-23256^2=\square$, $x^2-75000^2=\square$, $x^2-103224=\square$.
   Ans. $x=105625$.
5. Make $x+y+z=a^3$, $x+y=b^3$, $x+z=c^3$, and $y+z=d^3$ in positive whole numbers.

*I beg leave to state that on the 2nd day of March last, I received from this gentleman correct solutions to the questions 1, 2, 3, 4, 5, 10, 13, and 14. This I suppose is about his *ne plus ultra*—beyond it I defy him to advance.

" 'Till riper age shall with nature's force burnish his mind."

6. Make $x^2 \pm 96x = \square$, $x^2 \pm 135x = \square$ and $x^2 \pm 154x = \square$.

7. Make $182^2x^2 \pm 182x = \square$, $560^2x^2 \pm 560x = \square$ and $630^7x^2 \pm 630x = \square$.

8. Make $136^2x^2 \pm 136x = \square$, $170^2x^2 \pm 170x = \square$, and $174^2x^2 \pm 174x = \square$.

9. Make $(m^2+n^2)^2x^2 \pm (m^2+n^2)x = \square$, $(m^2-n^2)x^2 \pm (m^2-n^2)x = \square$, and $4m^2n^2x^2 \pm 2mnx = \square$.

10. Make $\dfrac{xyz}{x+y+z} = \square$, $x^2 + \dfrac{xyz}{x+y+z} = \square$, $y^2 + \dfrac{xyz}{x+y+z} = \square$,

    and $z^2 + \dfrac{xyz}{x+y+z} = \square$, all in whole numbers.

11. Make $(m^2+a^2)^2 \pm 4abm(m^2+a^2) = \square$, $m^2 - a^2 = \square$, and $(m^4+m^2a^2+a^4) \pm 4acm(m^2+a^2)(m^2-a^2) = \square$.

12. Make $x^2+y^2 = \square$, ${}^5/_4(x^2+y^2) =$ a cube, $xy = 2x^3$,

    $2(x+y) + \dfrac{xy}{x+y} = \square$, and $(x^4+y^4)(x^2+y^2)$

    $-(x^5+y^5)\sqrt{x^2+y^2 = \square}$.

13. Make $x^2 \pm 97104 = \square$, $x^2 \pm 150000 = \square$, $x^2 \pm 173400 = \square$, and $x^2 \pm 180576 = \square$.

14. Find the least values of x and y in whole numbers that solve the equation $x^2 - 940751xy^2 = 38$.

A certain teacher in this city who goes by the nick name of *Professor* has offered to solve the 9th question for \$5.00; I now offer him \$20.00 to prove either its possibility or *impossibility*, and show all the conditions that can exist and those that cannot.

Query—Will Professor James Renawick favor us with an explanation of the last 20 lines, Book I, page 8 of his excellent "Elements of Mechanics" just published. New York, June 1st, 1832.

Yours very respectfully,

JOHN D. WILLIAMS.

In reference to these problems, Dr. David S. Hart of Stonington, Conn., a gentleman who, in his day, solved many very difficult prob-

lems in Diophantine Analysis says in *Educational Notes and Queries*, Vol. 2, pages 71-72.:

"The 6th, 7th, 8th, 9th, 12th, and 14th are impossible of solution. The answers given to the first problem are much larger than those which I have found. Problems 3, 4, and 13 can be solved by one general method, which, may be, applies to the problem: 'To find a common value of x and also a, b, c, d, etc., which will make $x^2 \pm ax = \Box$, $x^2 \pm bx = \Box$, $x^2 \pm cx = \Box$, $x^2 \pm dx = \Box$, etc., etc., *ad infinitum.*'

This problem was sent to the *Educational Times*, London, but no solution has appeared. Some years ago, I effected a solution by means of an infinite series, whose law of continuation I determined, as also the nth term, by means of which all possible cases can be solved. I may send it to the *Times* or to some periodical in this country."

"In an interview I had with John D. Williams in 1865, he admitted that problem 12 was impossible, but he insisted that the remaining five were possible. Now, Prof. Theodore Strong, Dr. James Mattison, and others, have proved that no common value of x can be found that will solve the positive and negative signs conjunctively in 6, 7, 8, and 9."

Mr. J. D. Williams himself was an occasional contributor to the *Mathematical Diary* but we failed to find his name in either the *Mathematical Miscellany* or the *Cambridge Miscellany of Mathematics, Physics, and Astronomy*.

Dr. David S. Hart, Stonington, Conn., says that Williams evidently got up the *Companion* as a rival of the *Mathematical Diary*. "His opponents were numerous, and the contest was carried on with some bitterness, till finally Mr. Williams issued his 14 famous 'Challenge Problems' directed against all the mathematicians in America, excepting only Dr. Bowditch, Professor Strong, and Eugene Nulty. Six of these problems are impossible. Some of the others are somewhat difficult, but all have been solved by several persons."

Professor J. N. Michie, head of the department of mathematics at Texas Technological College, announces the appointment of Dr. Fred D. Rigby (University of Iowa) and Dr. Paul W. Gilbert (Duke University) as instructors for the academic year 1940-41.

The Report of the President for the academic year 1939-1940, as contained in the Official Register of Princeton University, includes (as is customary) a list of articles and reviews by members of the department of mathematics.

—Reported by L. J. Adams.

The Teacher's Department

Edited by
JOSEPH SEIDLIN and JAMES MCGIFFERT

The Trisection Problem

By ROBERT C. YATES
Louisiana State University

CHAPTER I

THE PROBLEM

1. The Famous Three

In the history of mathematics there are three problems that have persisted with astounding vigor for over two thousand years. They are *Trisecting the Angle, Duplicating the Cube,* and *Squaring the Circle,* and because of their hardy existence they are now called Famous Problems. The bare problems themselves, stripped of all implications, seem hardly worth more than passing attention and yet, even today, an incredible amount of energy is expended in the search for solutions by some means or other. We cannot help but wonder why three such apparently simple mathematical issues should stand forth above all others. Statements of the problems can be made in the simplest of terms and no one need be terrified by the heavy terminology usually associated with mathematical questions. It is just this disarming simplicity, however, that invites one to make courageous attacks. Doctors, lawyers, butchers and bakers, young men and old men, amateur mathematicians and professional ones, the sane and the insane—people in all walks of life have been drawn to them only to be snared insidiously in a web of their own spinning or to open for themselves suddenly and unexpectedly a path down which they could look into new fields. These three problems, solidly impregnable to all approaches from the vantage of plane geometry, the medium of the ancient Greeks, served only to tantalize and tease the mathematician

This is the first in a series of five chapters. The second will follow in an early issue.

into devising new apparatus and theorems for their solution. Through this stimulus did much of our present structure of algebra and geometry grow.

Constant search over so long a period for solutions of the Three Problems has yielded amazingly fruitful discoveries, often hit upon by the sheerest accident, that have thrown light in a totally unsuspected manner upon far distant things. The Ellipse, Parabola, and Hyperbola—sections formed from a cone by a cutting plane—are undoubtedly the most interesting and useful curves known. Without them we would be sore put to explain the heavens or to fire upon the hidden enemy or to peer into the habits of the microscopic world. It is said that these curves were discovered by Menæchmus in an ambitious attempt at the solution of the Three Problems. A further outgrowth was the development of that important field, the Theory of Equations. More indirectly, we find traces of their influence in the modern Group Theory, a doctrine of the highest importance to the physicist and chemist in their study of atomic structure and relativity theory. Little wonder then that these problems, to the credit of which so much mathematical activity is due, should now be classified as famous.

2. *A Classical Game*

The plane geometry of the ancient Greeks was a game to be played with simple equipment and governed by a rigid set of rules. The equipment consisted only of the *compasses* and an *unmarked straightedge,* indefinite in length. The rules, established and insisted upon by Plato,* were the postulates which allowed certain privileges in the use of the tools. These permitted:

1. The drawing of a straight line of indefinite length through two given distinct points;
2. The construction of a circle with center at a given point and passing through a second given point.

Indeed, it seems that a game built around such scanty outlay would be a disappointing affair. Nothing, however, could be farther from the truth. Probably the most fascinating game ever invented, it is awe-inspiring in its extent to the novice, and a thoroughly absorbing occupation to the expert.

Any geometry that was indulged in which did not adhere closely to the Platonian rules was condemned as unsportsmanlike and

*As rumor has it.

ill-befitting the ideal thoughts of the scholar and mathematician. This was the general opinion of the old classical school. All geometrical situations had to be met with only straightedge and compasses. But under their rules, these tools alone are incapable of producing solutions of the Three Problems. This fact, however, was not established until about 1800—two thousand years later. This statement is not surprising in view of the fact that it is necessary to pass beyond the confines of plane geometry in order to show that solutions cannot be found there. The mathematical structure needed to do this was a long time being developed and at first seemed to have nothing whatever to do with geometry.

3. *Trisection*

The first of the Three Problems, the trisection of a general angle, no doubt arose, so long ago that historians can find no record, in just the manner that we would propose it for ourselves today. We find it easy enough to *bisect* any angle whatever: with the compasses, locate a point which is equidistant from the sides of the given angle and then draw with the straightedge the line joining this point with the vertex of the angle. Success is easily won and we turn naturally to the division of the angle into three equal parts. After a variety of attempts restricted to the classical rules and tools the difficulties seem discouraging. We begin to suspect the existence of some underlying principles that block every move.

Hippias of Elis, who lived in the Fifth century B. C., was one of the first to attempt to solve the Trisection Problem. The very same obstacles presented themselves to thwart his efforts but, freeing himself from the Platonian rules, he devised a curve called the ***Quadratrix***, to be discussed later, by means of which he was able to give an exact solution to the problem. But, we must understand, it was not achieved by straightedge and compasses alone.

Hippias was only one of the first to succumb to the charm of this perplexing question. A partial list of his followers will show you what a powerful pull it had upon the attention of the great and the near-great. Archimedes, Nicomedes, Pappus, Leonardo da Vinci, Dürer, Descartes, Ceva, Pascal, Huygens, Leibniz, Newton, Maclaurin, Mascheroni, Gauss, Steiner, Chasles, Sylvester, Kempe, Klein, Dickson—all of these, and hundreds more, attacked the problem directly or created the mathematics by which substantial advances could be made toward a full understanding of the situation.

4. *Statement of the Problem*

Let us express the requirements for solution of the Problem in analytical form.

(A) *Algebraic Formulation.* Given the angle $AOB = 3\Theta$, let us suppose one of the trisecting lines to be OT, Fig. 1, so that $TOB = \Theta$. Select an arbitrary length on OA as the unit distance and draw the

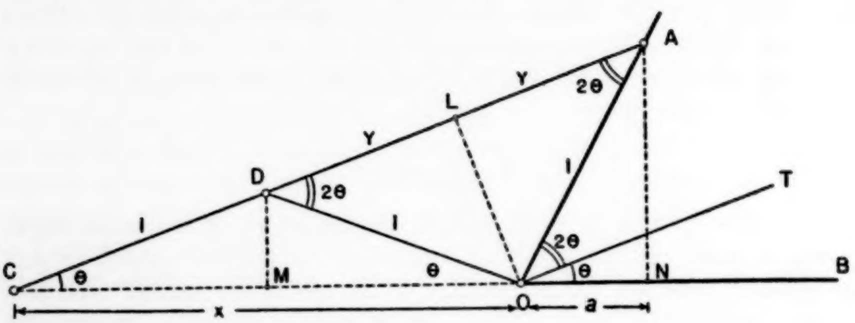

FIG. 1

parallel AC to OT, meeting OB extended at C. Then angle $DCO = \Theta$. Now draw OD equal to the unit length so that triangle AOD is isosceles with base angles 2Θ, angles DAO and AOT being equal since they are alternate interior angles. It is evident, since angle ADO is the sum of the opposite interior angles of triangles DCO and angle $DCO = \Theta$, that angle $DOC = \Theta$. Therefore, triangle DCO is isosceles and $DC = DO = 1$. Let x denote the distance OC, $2y$ the distance AD, and a the projection of OA upon the side OB. From similar triangles CMD, CNA, and CLO, all right triangles with equal angles at C, we find:

$$x/2 = (x+a)/(1+2y) = (1+y)/x,$$

which give $\qquad x^2 = 2+2y \quad \text{and} \quad 1+2y = 2(x+a)/x.$

From these we eliminate y by substitution to obtain:

$x^2 - 1 = \frac{2(x+a)}{x}$ $\qquad$ $$x^2+1 = 2(x+a)/x \quad \text{or}$$

(4.1) $$\boxed{x^3 - 3x - 2a = 0}\,.$$

This relation, as will be seen shortly, is fundamental to the problem and is called the Trisection Equation. It is a cubic equation with the term in x^2 missing.

(B) *Trigonometric Formulation.* A knowledge of trigonometry will produce the Trisection Equation in different fashion. In what follows we make use of the expression for the sine and cosine of the sum of two angles. We have:

$$\cos 3\theta = \cos(2\theta + \theta) = \cos 2\theta \cos \theta - \sin 2\theta \sin \theta.$$

This becomes, on replacing $\sin 2\theta$ and $\cos 2\theta$ by their equivalent values in terms of θ:

$$\cos 3\theta = (2\cos^2\theta - 1)\cos\theta - (2\cos\theta - 2\cos^3\theta), \text{ or}$$

$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta. \tag{4.2}$$

This expresses the cosine of a given angle in terms of the cosine of its third part. Looking again at Fig. 1:

$$x = 2\cos\theta \quad \text{and} \quad a = \cos 3\theta,$$

so that by making these replacements, (4.2) becomes:

$$a = x^3/2 - 3x/2,$$

or

$$x^3 - 3x - 2a = 0.$$

Note, before passing on, that no matter what angle is given, the corresponding value of a lies between $+1$ and -1 while that of x lies between $+2$ and -2.

Since we may drop the perpendicular from A upon OB and thus determine a, then we may think of this quantity as being *given* with the angle AOB. If the point C, or its distance x along OB, can be determined, the problem is at once solved by connecting C to A and then constructing the trisecting parallel OT. Thus we see that the *geometrical* solution of the problem is entirely equivalent to the *algebraic* solution of the corresponding Trisection Equation.

5. *Constructibility*

We may now restate the proposal in a different way: *Is it possible, for all values of a, to find by a straightedge and compasses construction the root x of the Trisection Equation?* The answer, suspected for so long, that it is not always possible is now definitely established.

Any construction which depends on the location of points by means of the straightedge and compasses is a permissible one under the rules of plane geometry. To conserve space we shall use the word *constructible* for the operations that can be performed with these tools. Since the Trisection Problem has now been put upon an algebraic footing, we must see how these operations appear in algebraic form.

For, it is only through this medium that we can determine the character of the solution we seek.

(A) *Algebraic Equivalence of Constructibility.* If we are given two line segments, a and b, these segments can be added, subtracted, multiplied, and divided geometrically, using only straightedge and compasses. These operations on the given quantities are called *rational.* The first two need no explanation and are evident from the meanings of the words *sum* and *difference.* The multiplication of a and b is effected by drawing the line PQ, Fig. 2A, at any angle with PR and constructing the parallel line to produce similar triangles as

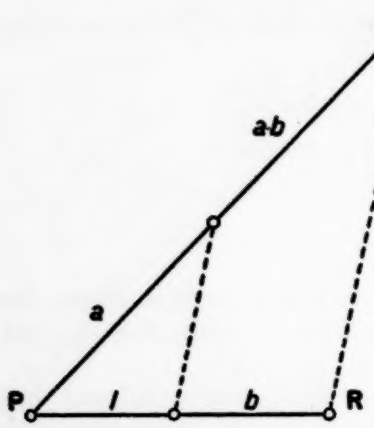

FIG. 2A

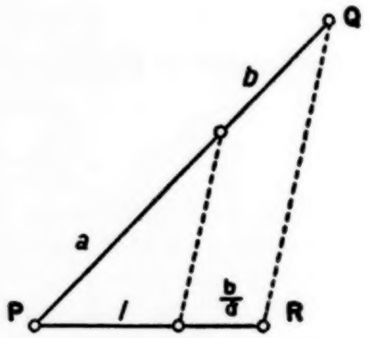

FIG. 2B

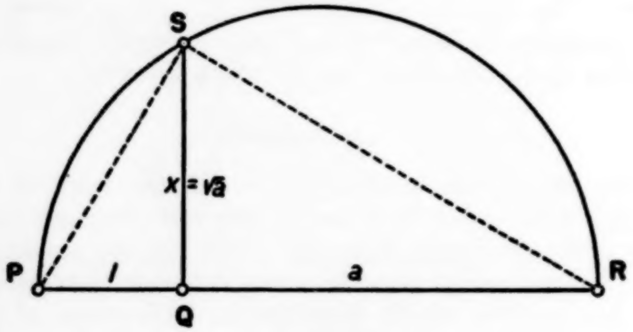

FIG. 2C

shown. Since a and b represent the *ratio* of segment length to unit length, the unit is given with a and b. Division of the segments, b/a, is similar to multiplication and should be obvious from Fig. 2B. The

extraction of a root of a quantity is called an *irrational* operation. The *square root* of a line segment, a, may be constructed by drawing the semicircle with $(1+a)$ as a diameter and erecting the perpendicular at the joint of the segment and the unit distance. It should not be difficult to see from the two similar right triangles the proportion: $1/x = x/a$, and thus $x = \sqrt{a}$.

We shall presently show that these five algebraic operations are the only ones that admit construction by straightedge and compasses. Combinations of these, however, can be built up step by step to produce very complicated constructions. For example,

$$(1)\quad a+b\sqrt{c} \qquad\qquad (2)\quad \sqrt{(a+\sqrt{b+\sqrt{c}})}$$

$$(3)\quad (a+\sqrt{b})/(c+\sqrt{d}) \qquad\qquad (4)\quad \sqrt{a}+\sqrt{(b+\sqrt{c})}$$

are all constructible if the quantities a, b, c, d are given lengths and no imaginaries appear. Thus, for (2) we would first take the square root of c, then add b, then take the square root of the result, then add a, and finally take the square root of that result—all accomplished by straightedge and compasses as shown in Figure 2.

Generally, such expressions are called *quadratic irrationalities of order n*, where n is the least possible number of superimposed square root radicals. Number (3), for instance, is of order 3. Complicated as these irrational quantities appear, it will be noticed that they involve nothing more than a series of square roots of constructible lengths and they are, therefore, themselves constructible. We shall use a general symbol to represent all of them:

$$A+B\sqrt{C},$$

where A, B, C are constructible quantities and, generally, $\sqrt{C}$ is a quadratic irrationality of higher order than A and B.

Numbers of this sort may be roots of equations of much higher degree than the second—equations whose coefficients are either the given lengths or rational functions of them. Let us take a single illustration from the preceding group. If we set, for (2):

$$x = (a+\sqrt{b})/(c+\sqrt{d}),$$

and square, we have:

$$c^2x^2-2acx+a^2 = b-2x\sqrt{(bd)}+dx^2.$$

Collecting and squaring again to remove the radical $\sqrt{(bd)}$:

$$(c^2-d)^2x^4-4ac(c^2-d)x^3+(6a^2c^2-2bc^2-2a^2d-2bd)x^2 \\ -4(a^2-b)acx+(a^2-b)^2=0,$$

an equation of fourth degree in x whose coefficients are rational functions of a, b, c, and d.

We shall now prove that the rational operations of addition, subtraction, multiplication, and division, together with the irrational operation of extraction of square roots are the only ones possible by straightedge and compasses. To this end we transfer to analysis and use the algebraic interpretation of these geometrical elements.

All constructions of plane geometry are but the location of points either as the intersection of two lines, a line and a circle, or two circles.

I. Two given or constructed lines are represented by the equations

$$a_1x+b_1y+c_1=0$$
$$a_2x+b_2y+c_2=0$$

where the coefficients are geometrical lengths either given to start with or determined at some stage in the construction. These lines intersect in the point whose coordinates are the simultaneous solution of their equations, that is, in

$$x=(b_1c_2-b_2c_1)/(a_1b_2-a_2b_1), \quad y=(a_1c_2-a_2c_1)/(a_1b_2-a_2b_1).$$

These numbers are evidently rational functions of the coefficients in the equations of the lines. Thus the manipulation of the straightedge leads to no operation other than addition, subtraction, multiplication, and division of lengths.

II. The line $$ax+by+c=0$$

meets the circle $$(x-h)^2+(y-k)^2=r^2$$

in points whose abscissas are given by

$$(a^2+b^2)x^2+2(ac-hb^2+abk)x+c^2+2bck+(h^2+k^2-r^2)b^2=0,$$

or $$Ax^2+Bx+C=0,$$

where the coefficients A, B, C are clearly rational functions of the quantities a, b, c and h, k, r. The solutions of this quadratic are

$$x=(-B\pm\sqrt{B^2-4AC})/2A,$$

quantities which involve, in addition to the operations mentioned in I, nothing further than the constructible irrationality $\sqrt{B^2-4AC}$.*

III. The intersections of two circles are the same as the intersections of their common chord and one of the circles. Thus, since

*If the quantity B^2-4AC is negative, the root is imaginary and there is no question of constructibility since the line and circle do not meet.

the coefficients in the equation of the chord are rational functions of those in the equation of the circles, this case reduces immediately to II. Accordingly,

The straightedge and compasses together are capable of making only those geometrical constructions which are algebraically equivalent to a finite number of the operations of addition, subtraction, multiplication, division and the extraction of real square roots involving the given lengths.

6. *The Impossible*

In order to determine the impossiblity of trisection of the general angle, it suffices to give but a single example. Returning to the Trisection Equation, we shall discuss the situation for the particular angle $AOB=60°$. For this the projection value $a=\cos 60°=1/2$ produces the corresponding equation:

$$x^3-3x-1=0. \tag{6.1}$$

The question that must be decided is whether or not this equation has constructible roots of the sort described in the preceding pages. If not, then trisection is not always possible by straightedge and compasses. The argument is a bit involved, to be sure, but the end in view is worth the effort.

In order to proceed without interruption, we shall dispose of a necessary preliminary consideration. If x_1, x_2, x_3 are the roots of (6.1) we may write the equation as

$$(x-x_1)(x-x_2)(x-x_3)=0,$$

or

$$x^3-(x_1+x_2+x_3)x^2+(x_2x_3+x_3x_1+x_1x_2)x-x_1x_2x_3=0.$$

This, however, is identical with

$$x^3-3x-1=0.$$

It is evident on comparing these two forms that the sum of the roots of equation (6.1) is zero. That is, since the term in x^2 is missing, its coefficients must be zero. Thus

$$x_1+x_2+x_3=0. \tag{6.2}$$

(A) We shall first prove that *(6.1) does not have a rational root.* If we assume that it does, we are led to a contradiction, as follows:

Let $x=A/B$, where A,B are integers with no common factor other than 1. Then from (6.1): $(A/B)^3-3(A/B)=1$, which may be written in either of the forms:

$$3A+B=A(A/B)^2 \quad \text{or} \quad A^2-3B^2=B^2(B/A).$$

Now, since A and B are both integers, the left hand member of each of these equations is an integer. Accordingly, the right hand members must be integers and, since neither A nor B has any common factors other than 1, the only possibility is that A and B have either of the values $+1$ or -1. That is, $x=\pm 1$. But this is impossible for neither $+1$ nor -1 satisfies (6.1). This establishes the statement that $x^3-3x-1=0$ has no rational root.

(B) If (6.1) has a constructible root of the sort $x_1=A+B\sqrt{C}$ where A and B are constructible irrationalities of lower order than $\sqrt{C}$, then on substituting in (6.1) we have:

$$(A+B\sqrt{C})^3-3(A+B\sqrt{C})-1=0,$$

or

$$(A^3+3AB^2C-3A-1)+(3AB+B^3C-3B)\sqrt{C}=0.$$

The only condition under which this can exist is that both quantities in the parentheses be zero. But this implies something further. Since the substitution of $(A-B\sqrt{C})$ produces the same equation except for a change in sign between the parentheses, we are apparently in possession of a second root: $x_2=A-B\sqrt{C}$. But, from (6.2) the three roots have zero for their sum; that is,

$$x_1+x_2+x_3=A+B\sqrt{C}+A-B\sqrt{C}+x_3=0,$$

or

$$x_3=-2A.$$

If, as we supposed, A is a constructible irrationality it must be of the sort, $L+M\sqrt{N}$ with $\sqrt{N}$ of higher order than either L or M but yet of lower order than $\sqrt{C}$. A repetition of the preceding argument applies here and forces us to admit the existence of a root whose irrationality is of the same order as L. Thus we are led from link to link down this chain of reason until we find the only constructible root that this equation *might* have is a rational number. But we demonstrated in (A) that it did not have such a root. Therefore, equation (6.1) has no constructible root and

60° cannot be trisected by straightedge and compasses.

7. *The Possible*

From the preceding discussions it is evident that certain angles do admit of trisection by straightedge and compasses. In fact, if the Trisection Equation

$$x^3-3x-2a=0 \tag{7.1}$$

can be factored into the form:

$$(x+r)(x^2+sx+t)=0 \tag{7.2}$$

where r, s, t are constructible coefficients, then the angle whose cosine is a can be trisected by these means. Since (7.1) and (7.2) are here identical, we may equate their coefficients, having:

$$r=-s, \quad t=r^2-3, \quad rt=-2a,$$

and the three roots of (7.2) may be written as:

$$x_1=-r; \quad x_2=(1/2)\left(r+\sqrt{3}\sqrt{1+2a/r}\right); \quad x_3=(1/2)\left(r-\sqrt{3}\sqrt{1+2a/r}\right).$$

To illustrate such a possibility, consider the given angle $AOB=54°$ whose cosine is $(1/4)\sqrt{(10-2\sqrt{5})}$. The corresponding Trisection Equation is therefore

$$x^3-3x-(1/2)\sqrt{(10-2\sqrt{5})}=0,$$

which can be factored into the forms:

$$[x-(1/4)(\sqrt{5}+1)\sqrt{10-2\sqrt{5}}]=0$$

$$[x^2+(x/4)(\sqrt{5}+1)\sqrt{10-2\sqrt{5}}+(\sqrt{5}-1)/2]=0.$$

Notice that all coefficients, complicated as they are, are constructible and all roots are consequently constructible. Thus 54° can be trisected, or, which is the same thing, 18° can be constructed by straightedge and compasses.

The *discriminant*, D, of a cubic equation is an expression which indicates the character of its roots. For the Trisection Equation this discriminant is the quantity:

$$D=108(1-a^2).$$

Now, since a cannot be greater than 1, D is always positive and this assures us that all three roots of the equation are real numbers. Why should there be three when only one is all that is necessary to be determined for a given angle? The answer is found in realizing that the quantity a is not only the cosine of the given angle, 3θ, but also of $(360°+3\theta)$ and of $(720°+3\theta)$. Accordingly, the Trisection Equation delivers to us a root which determines the trisection of the given angle and two further "induced" roots corresponding to the angles $(120°+\theta)$ and $(240°+\theta)$. Thus for $3\theta=90°$, $a=\cos 90°=0$, and the Trisection Equation $x^3-3x=0$ produces the three roots: $+\sqrt{3}$, $-\sqrt{3}$, and 0. The first of these values corresponds to the third part, 30°, of the given angle. The two remaining values give constructions for 150° and 270° as the third parts of the two induced angles.

Some Trisection Equations belonging to familiar angles which fall under the "possible" case are listed in the accompanying table:

AOB	$a=\cos(AOB)$	Trisection Equation	Roots
0°	1	$x^3-3x-2=0$	$-1,\ -1,2$
45°	$\sqrt{2}/2$	$x^3-3x-\sqrt{2}=0$	$-\sqrt{2},\ (\sqrt{2}/2)(1\pm\sqrt{3})$
72°	$(\sqrt{5}-1)/4$	$x^3-3x-(\sqrt{5}-1)/2=0$	$-2/(\sqrt{5}-1),1/(\sqrt{5}-1)\pm\sqrt{\dfrac{15-6\sqrt{5}}{2(3-\sqrt{5})}}$
90°	0	$x^3-3x=0$	$0,\ +\sqrt{3},\ -\sqrt{3}$
180°	-1	$x^3-3x+2=0$	$1,\ 1,\ -2$

Since we can trisect 72° and can bisect *any* angle, it follows that an angle of 3° is constructible. On the other hand, angles of 1° and 2° are not constructible for, otherwise, we would be able to trisect 60°. It is somewhat startling to realize that the unit of angular measure we have used with so much familiarity cannot be constructed with straightedge and compasses.

8. *Other Criteria*

Although it is impossible to give a simple criterion to apply to all angles, the following discussion leads to rules that produce an infinitude of possibilities.

(A) If n is a given integer *not a multiple of 3*, then the equation

$$n\cdot b+3\cdot c=1 \tag{8.1}$$

can always be satisfied by finding particular integer values for b and c. Thus, for example, $4b+3c=1$ is satisfied by $b=4$, $c=-5$; or $b=-5$, $c=7$; etc.; $-13b+3c=1$ by $b=2$, $c=9$; or by $b=-1$, $c=-4$. Multiplying (8.1) throughout by $(360°/3n)$, we have:

$$b(120°)+c(360°/n)=(1/3)(360°/n). \tag{8.2}$$

Now if the given angle AOB is of this type, $360°/n$, (18° for example), then (8.2) may be written (reversing the order):

$$AOB/3=c(AOB)+b(120°).$$

The angle 120° is itself constructible and we can always find integers b and c to satisfy this last equation. Thus, to construct $AOB/3$, we need only multiply the given angle by c, the angle 120° by b, and add

the result—all of which are possible constructions. It should be clear then that

If $AOB = 360°/n$, where n is an integer not divisible by 3, then AOB admits of trisection by straightedge and compasses.

Obviously, $k(360°/n)$ is an angle in the same class if k is an integer.

(B) Suppose now that $AOB = 360°/n$ where n is a multiple of 3, say $n = 3^r \cdot m$ where 3^r contains all the factors 3, and m does not contain any. Then, as in the preceding, two integers b and c can be found such that

$$mb + 3c = 1$$

is satisfied. Multiplying this last equation through by $(360°/3n)$, we obtain:

$$mb(360°/3n) + c(360°/n) = (1/3)(360°/n).$$

In the first term, however, $m/n = 1/3^r$, and thus

$$b(120°/3^r) + c(360°/n) = (1/3)(360°/n)$$

or

$$AOB/3 = c(AOB) + 2b(60°/3^r).$$

Now since r is a positive integer, the last term in the right member is either 20° or some repeated trisection of 60°. We have shown that 60° cannot be trisected by straightedge and compasses and it follows that $60°/3^r$ is not a constructible angle. Accordingly, $AOB/3$ is not constructible in this case and a companion rule to the preceding one is established:

If $AOB = 360°/n$, where n is an integer divisible by 3, then AOB cannot be trisected by straightedge and compasses.

Another set of each class may be determined by the two following rules:

If p and q are integers and p is numerically less than q then it is possible to trisect by straightedge and compasses any angle whose cosine is

$$a = (p^3 - 3pq^2)/2q^3.$$

For, the corresponding Trisection Equation:

$$x^3 - 3x - (p^3 - 3pq^2)/q^3 = 0$$

is obviously satisfied if $x = p/q$, and this root is constructible. An example is furnished by the values $p = -1$, $q = 3$. For these,

$\cos(AOB) = 13/27$ and AOB is approximately $61°13'$. On the other hand,

If the cosine of the given angle is p/q, where p and q are integers without common factors and q is greater than 1 but not the multiple of a cube, then it is impossible to trisect this angle by straightedge and compasses.

9. *Regular Polygons*

The general question of trisection enters into the study of the possibility of constructing regular polygons. Those of three, four, five, six, ten, and fifteen sides, for example, are constructible by straightedge and compasses, a fact well known to the ancient Greeks. But the polygons of seven, nine, eleven sides cannot be so constructed. This fact, like the proof of the impossibility of general trisection, was also late in being established. The ennagon, or 9-sided polygon, has the central angle of 40° subtended by each side and we have seen that this angle is not constructible. The construction of the 7-sided polygon depends on an equation of the third degree which can be shown, by a treatment similar to that of Paragraph 6, to contain no constructible roots. Gauss was the first to give a general constructibility rule for all regular polygons thus bringing to light some possibilities that were never dreamed of up to that time. Among the constructible ones were found the polygons of 17, 257, and even 65,536 sides. Unfortunately, the scope of this book does not permit us to wander down this enchanting path.

Professor L. P. Eisenhart (Princeton University) has been reelected Chairman of the Division of Physical Sciences of the National Research Council for a period of three years.

Dr. Claude Chevalley, a graduate of the École Normale, has been appointed assistant professor at Princeton University.

Officers of the mathematical section of the Southwestern Division of the American Association for the Advancement of Science for the year 1940-41 include: Chairman, Dr. E. J. Purcell, University of Arizona, Tuscon; Vice-chairman, Dr. Roy MacKay, Eastern New Mexico College, Portales; Secretary, Dr. Harold Larsen, University of New Mexico, Albuquerque. Dr. C. V. Newsom, department of mathematics, University of New Mexico, was elected president of the Southwestern Division for the year 1940-41. The 1941 meeting will be held in Lubbock, Texas.

—Reported by L. J. Adams.

Mathematical World News

Edited by
L. J. ADAMS

The program of the seventh December meeting of the National Council of Teachers of Mathematics is given in detail in the November, 1940 number of *The Mathematics Teacher*. This meeting, to be held on December 30, 1940-January 1, 1941, at Baton Rouge, Louisiana, will include some fifteen separate meetings distributed over Monday, Tuesday and Wednesday.

As is customary, the September, 1940 number of the *Bulletin* of the American Mathematical Society contains a list of officers and members for 1939-1940. The September number is issued in two parts, and this list is in part 2.

Professor G. D. Birkhoff of Harvard University has been elected Foreign Associate of the National Academy of Sciences of Lima.

The following news items are from the University of Oklahoma:

S. W. Reaves, head of the mathematics department at the University of Oklahoma since 1905 and dean of the College of Arts and Sciences since 1923, has resigned from his administrative duties but is still professor of mathematics. E. D. Meacham, professor of mathematics and assistant dean of the College of Arts and Sciences, was made dean of that college; J. O. Hassler, also a professor of mathematics, was made head of the mathematics department. E. P. R. Duval has been promoted from associate professor to professor of mathematics. W. C. Randels has been promoted from assistant professor to associate professor of mathematics. B. S. Whitney was appointed instructor in mathematics and astronomy. Commander J. C. Van de Carr, U. S. N., retired, has been recalled to active duty in the navy and placed in charge of the University of Oklahoma unit of the Naval R. O. T. C., having been granted a leave of absence as instructor in mathematics. J. C. Brixey, S. B. Townes, C. E. Springer, and Dora McFarland, assistant professors, were promoted to associate professors; R. D. Dorsett, instructor, became assistant professor.

The National Council of Teachers of Mathematics will meet at the Louisiana State University in Baton Rouge, Louisiana, from December 30 to January 1, inclusive. The theme of the meeting is *The*

Relationship between Enriched Mathematics Experience and Enriched Community Experience. Miss Mary Potter is President of the National Council.

Samuel I. Jones of Nashville, Tennessee announces the publication of a new book *Mathematical Clubs and Recreations.*

Miss Harriet E. Glazier, University of California at Los Angeles, has retired, with the title of assistant professor emeritus.

The general chairman for the War Preparedness Committee of the American Mathematical Society and the Mathematical Associaiation of America is Professor Marston Morse. The chairman of the subcommittee on Preparation for Research is Professor M. H. Stone. The chairman of the subcommittee on Education for Service is Professor W. L. Hart.

Professor Lancelot Hogben, author of *Mathematics for the Million,* is a member of the faculty at the University of Wisconsin for the year 1940-41. Professor Hogben is giving a course on the history of science.

Professor V. Volterra, University of Rome, died on October 11, 1940.

Dr. C. C. MacDuffee has been appointed professor of mathematics in Hunter College, New York City.

The *Science News Letter* for October 19, 1940 contains a list of new books on science, published or to be published between July 1 and December 31, 1940. One subdivision of the list is devoted to books on mathematics.

Professor George Polya, Technische Hochschule (Zurich), has been appointed member of the department of mathematics at Brown University.

Application blanks for the annual stipends, awarded several mathematicians with preparation near the Ph. D. level, may be obtained from the School of Mathematics, Institute for Advanced Study, Fuld Hall, Princeton, New Jersey. The application blanks are returnable by February 1, 1941.

Professor Glenn James, University of California at Los Angeles, is on sabbatical leave for the current semester.

Problem Department

Edited by
ROBERT C. YATES and EMORY P. STARKE

This department solicits the proposal and solution of problems by its readers, whether subscribers or not. Problems leading to new results and opening new fields of interest are especially desired and, naturally, will be given preference over those to be found in ordinary textbooks. The contributor is asked to supply with his proposals any information that will assist the editors. It is desirable that manuscript be typewritten with double spacing. Send all communications to ROBERT C. YATES, Mathematics, University, Louisiana.

SOLUTIONS

No. 346. Proposed by *E. C. Kennedy*, Texas College of Arts and Industries.

It is required to determine a function, $f(z)$, of $z=x+iy$ subject to the following conditions. Put

$$\varphi = \lim_{z\to 0} \frac{f(z)-f(0)}{z},$$

where z approaches 0 along the path $x=y^k$. Then for $0<k<1$, $\varphi=0$; for $k=1$, $\varphi=1$; $1<k<2$, $\varphi=\infty$; $k=2$, $\varphi=1$; $k>2$, $\varphi=0$.

Solution by the *Proposer*.

Write $f(z)/z=N/D$. A little preliminary experimentation suggests the expression $N/D=y^{k-2}/(y^{3k-4}+1)$. Whence one value is given by

$$f(z)=\frac{xy^2(x+iy)}{x^3+y^4}, \quad z\neq 0; \quad f(0)=0.$$

The solution is not unique. Another value is

$$f(z)=x^2y^2(x+iy)/(x^4+y^6).$$

No. 347. Proposed by *Paul D. Thomas*, Norman, Oklahoma.

Construct a triangle given a side, the difference between an adjacent angle and the Brocard angle, and the distance of the Brocard point from the given side.

Solution by *Davis P. Richardson*, University of Arkansas.

Given side AB; distance d from AB to Brocard point Q; and $\angle D = \angle A -$ Brocard angle α. Draw AB and a line x, d units from it. At A construct $\angle BAR = \angle D$ on the same side of AB with x. AR meets x in the Brocard point Q. Draw QB. Then $\angle QBA = \alpha$, the Brocard angle. Draw AK so that $\angle QAK = \alpha$. Through Q and B draw a circle such that QB subtends the angle α at each point of the circumference. The intersections of the circle with AK are the two positions for the third vertex C. There are thus two solutions.

Also solved by *D. L. MacKay* and the *Proposer*.

No. 350. Proposed by *D. L. MacKay*, Evander Childs High School, New York.

Show that 5525 is the hypotenuse of twenty-two integral right triangles. Find them.

Solution by *G. W. Wishard*, Norwood, Ohio.

We need two well-known propositions from the theory of numbers:

(a) The sides of every integral right triangle are given by the formulas:

$$a = 2kxy, \qquad b = k(x^2 - y^2), \qquad c = k(x^2 + y^2),$$

where x and y have no common factor, one of them is even, and $x > y$.

(b) A product $P = LM$ can be represented as a sum of two squares if each factor can, viz.

$$(r^2+s^2)(u^2+v^2) = (ru+sv)^2 + (rv-su)^2. \tag{1}$$

Conversely every representation of P as a sum of two squares can be obtained from representations for L and M by use of (1).*

Now $5525 = 5^2 \cdot 13 \cdot 17 = k(x^2+y^2)$, $5 = 2^2+1^2$, $5^2 = 3^2+4^2$,

$$13 = 2^2+3^2, \quad 17 = 4^2+1^2,$$

whence various factorizations of 5525 and repeated application of (1) give the required twenty-two sets as follows:

*See, for example, Carmichael, *Diophantine Analysis*, pp. 10, 24, ff.

k	x^2+y^2	x		a	b	k	x^2+y^2	x	y	a	b
1105	5	2	1	4420	3315	13	425	16	13	5408	1131
425	13	3	2	5100	2125			19	8	3952	3861
325	17	4	1	2600	4875	5	1105	32	9	2880	4715
221	25	4	3	5304	1547			31	12	3720	4085
85	65	8	1	1360	5355			33	4	1320	5365
		7	4	4760	2805			24	23	5520	235
65	85	9	2	2340	5005	1	5525	74	7	1036	5427
		7	6	5460	845			73	14	2044	5133
25	221	11	10	5500	525			71	22	3124	4557
		14	5	3500	4275			62	41	5084	2163
17	325	17	6	3468	4301						
		18	1	612	5491						

Also solved by *C. C. Chaudoir, Edwin Comfort, Dewey C. Duncan, Frank H. Mehrhoff, C. W. Trigg,* and the *Proposer.*

No. 351. Proposed by *M. S. Robertson,* Rutgers University.

If m is a non-negative integer, find the sum function for

$$\sum_{n=1}^{\infty} \frac{(n+z)^m}{(n+1)!} .$$

Solution by *Margaret Woods,* Evanston, Illinois.

Set
$$S_m(z) = \sum_{n=1}^{\infty} \frac{(n+z)^m}{(n+1)!} .$$

Then for $m=0, 1, 2, \cdots$, we have the following:

$$S_{m+1}(z) = \sum_{n=1}^{\infty} \frac{(n+z)^{m+1}}{(n+1)!} = \sum_{n=1}^{\infty} \frac{(n+z)^m}{(n+1)!}(n+1+z-1)$$

$$= \sum_{n=1}^{\infty} \frac{(n+z)^m}{n!} + (z-1)\sum_{n=1}^{\infty} \frac{(n+z)^m}{(n+1)!}$$

$$= \sum_{n=0}^{\infty} \frac{(n+1+z)^m}{(n+1)!} + (z-1)\sum_{n=1}^{\infty} \frac{(n+z)^m}{(n+1)!} ,$$

which is equivalent to

$$S_{m+1}(z) = (z+1)^m + S_m(z+1) + (z-1)S_m(z).$$

We have also

$$S_0(z) = \sum_{n=1}^{\infty} \frac{1}{(n+1)!} = \sum_{n=2}^{\infty} \frac{1}{n!} = e-2,$$

from which the polynomials $S_m(z)$ may be computed in succession.

Also solved by *E. C. Kennedy*, and the *Proposer*, who shows that the polynomials $S_m(z)$, $m=1, 2, 3, \cdots$, are of the form

$$\sum_{r=0}^{m} \binom{m}{r} A_r z^r$$

in which A_r is the value for $z=0$ of the $(m-r)$th derivative of

$$(e^{e^z-z} - e^{-z} - 1).$$

It is then easy to see that $A_m = e-2$, $A_{m-1}=1$, and each other A_r is of the form $Pe \pm 1$ where P is a positive integer. Can a closed expression be found for P as a function of $(m-r)$?

No. 353. Proposed by *Paul D. Thomas*, Norman, Oklahoma.

Given any homogeneous polynomial $f(x,y)$ of degree n with real coefficients which satisfies the Laplace equation $f''_x + f''_y = 0$; show that $f(x,y)=0$ represents n lines which makes angles of π/n with one another.

Solution by the *Proposer*.

With $$f(x,y) = \sum_{r=0}^{n} \binom{n}{r} a_r x^{n-r} y^r$$

the hypothesis becomes

$$f''_x + f''_y = \sum_{r=0}^{n-2} [n!/(n-r-2)!r!] a_r x^{n-r-2} y^r$$

$$+ \sum_{r=0}^{n-2} [n!/(n-r-2)!r!] a_{r+2} x^{n-r-2} y^r = 0,$$

or $$a_r + a_{r+2} = 0, \quad r=0, 1, 2, \cdots, n-2.$$

This condition and the substitution $x = R\cos\Theta$, $y = R\sin\Theta$ reduce $f(x,y)$ to the form,

$$f(x,y) = a_0 R^n \sum_{k=0}^{(n/2)} (-1)^k \binom{n}{2k} \cos^{n-2k}\Theta \sin^{2k}\Theta$$

$$+a_1R^n\sum_{k=0}^{[(n-1)/2]}(-1)^k\binom{n}{2k+1}\cos^{n-2k-1}\Theta\,\sin^{2k+1}\Theta$$

$$=a_0R^n\cos n\Theta+a_1R^n\sin n\Theta.^*$$

Thus $f(x,y)=0$ reduces to

$$\tan n\Theta=-a_0/a_1 \quad \text{or}$$

$$\arctan y/x=\Theta=[\arctan(-a_0/a_1)+m\pi]/n, \quad m=0,1,2,\cdots,n-1,$$

which is the analytic statement of the desired conclusion.

No. 354. Proposed by *Walter B. Clarke*, San Jose, California.

Construct a scalene triangle having a median, an altitude and an external angle bisector concurrent.†

Solution by *E. C. Kennedy*, Texas College of Arts and Industries.

Let $DABC$ be a straight line with B the midpoint of AC. Construct a circle with radius AB and center at B. Take any point P on the circle. Draw AP. Extend CP until it strikes the bisector of angle DAP at Q. Draw QB, cutting AP at E. Then AEC is obviously the required triangle.

Also solved by *Paul D. Thomas* and *D. L. MacKay*.

No. 355. Proposed by *V. Thébault*, Le Mans, France.

Show that the three-digit number 111 is not a perfect square in any system of numeration. Is the same true of the five-digit number 11111 ?

Solution by the *Proposer*.

If the base of the system of numeration is represented by B, $B>1$, the number in question may be written $111=B^2+B+1$. Now

$$B^2<B^2+B+1<(B+1)^2.$$

Hence, whatever the value of B, B^2+B+1 is never a perfect square.

The same is not true of the number

$$11111=B^4+B^3+B^2+B+1.$$

*c. f. the identity, $\cos n\Theta+i\sin n\Theta=(\cos\Theta+i\sin\Theta)^n$.

†It is intended, of course, that the three lines should not be issued from the same vertex.—ED.

Indeed, if $B=3$, $11111=(102)^2$. That this is the only value of B appears from the following relations:

$$(B^2+\tfrac{1}{2}B)^2<B^4+B^3+B^2+B+1<(B^2+\tfrac{1}{2}B+1)^2,$$

$$(B^2+\tfrac{1}{2}B+\tfrac{1}{2})^2=B^4+B^3+B^2+B+1 \text{ implies } B=3.$$

No. 356. Proposed by *C. W. Trigg*, Los Angeles City College.

1. If a line be divided into n equal segments, the sum of the squares of the lines joining any point, P, to the extremities of the segments is equal to $(n+1)/2$ times the sum of the squares of the extreme joins diminished by $n(n^2-1)/6$ times the square of one of the segments.

2. From (1) show that the sum of the squares of the rays joining the vertex of the right angle to the points of n-section of the hypotenuse of a right triangle is equal to $(n-1)(2n-1)/6n$ times the square of the hypotenuse.

Solution by *D. L. MacKay*, Evander Childs High School, New York.

We employ a theorem which, though generally attributed to Stewart and given in Carnot's *Geometrie de position*, p. 263, was due to Robert Simson. See *Intermédiaire des mathématiciens*, 1908, pp. 160, 188: If in triangle ABC the line AD divides BC in two segments m and n, then $AD^2 \cdot BC=m\cdot AC^2+n\cdot AB^2-mn\cdot BC$.

(1) Let the line A_0A_n be divided into n equal parts, each equal to g and designate the lines PA_i by a_i, where $i=0, 1, 2, \cdots n$. Then

$$n\cdot a_j^2=ka_0^2+ja_n^2-njkg^2, \quad (j=1, 2, \cdots n-1), \quad (j+k=n).$$

$$n\sum a_j^2=a_0^2\sum k+a_n^2\sum j-ng^2\sum jk.$$

But $$\sum j=n(n-1)/2=\sum k$$

and $$\sum jk=n\sum j-\sum j^2=n^2(n-1)/2-n(n-1)(2n-1)/6.$$

Hence

(1) $$\sum a_j^2=(n-1)(a_0^2+a_n^2)/2-n(n^2-1)g^2/6.$$

Adding $(a_0^2+a_n^2)$ to each member, we have

$$\sum a_i^2=(n+1)(a_0^2+a_n^2)/2-n(n^2-1)g^2/6.$$

(2) If $\angle A_0PA_n$ is a right angle, (1) becomes

$$\sum a_j^2=(n-1)(A_0A_n)^2/2-n^2(n^2-1)g^2/6n=(n-1)(2n-1)(A_0A_n)^2/6n.$$

Also solved by the *Proposer* who refers to Court's *College Geometry*, p. 114, and to *School Science and Mathematics*, p. 938, November, 1938.

No. 359. Proposed by "*A Wag*", Cincinnati, Ohio.

	1	3	6	10	15	21	28	36	45	55	66	78
	1	4	9	16	25	36	49	64	81	100	121	144
	1	8	27	64	125	216	343	512	729	1000	1331	1728
3)	3	15	42	90	165	273	420	612	855	1155	1518	1950
	1	5	14	30	55	91	140	204	285	385	506	650

What does this mean?

Solution by *C. C. Chaudoir*, Baker, Louisiana.

Several observations may be made. (1) The sum of the first n squares in the second row is the nth number of the fifth row. (2) The sum of the first n cubes in the third row is the square of the nth number in the first row. (3) The meaning of the nth column is: The arithmetic mean of the nth triangle, the nth square and the nth cube is equal to the sum of the first n squares. In symbols

$$[\tfrac{1}{2}n(n+1)+n^2+n^3]/3=n(n+1)(2n+1)/6=1^2+2^2+3^2\cdots+n^2.$$

Also solved by *W. B. Clarke.*

No. 360. Proposed by *D. L. MacKay*, Evander Childs High School.

Given the sides of its two inscribed squares, construct the right triangle ABC.

Solution by the *Proposer*.

Let $CDEF$, D on BC, F on AC, and $GHKL$, G on BC, H and K on the hypotenuse AB, be the inscribed squares of the right triangle (a,b,c). Let h be the altitude on the hypotenuse, $DE=d$ and $GH=e$.

Then (1): $c^2=a^2+b^2$; (2): $ch=ab$; and from the similar triangles ABC and GCL,

$$c/e=h/(h-e) \quad \text{or} \quad h=ce/(c-e).$$

Substituting this value in (2) we have (3): $ab=c^2e/(c-e)$. In like manner from the similar triangles ABC and BDE,

$$b/d=a/(a-d)=(a+b)/a \quad \text{or} \quad (4)\colon ab=d(a+b).$$

Adding twice (3) to (1), we have (5): $(a+b)^2=c^2(c+e)/(c-e)$. Substituting (3) and (5) in the square of (4), we have:

$$c^2e^2=d^2(c^2-e^2) \quad \text{or} \quad c=ed/\sqrt{(d^2-e^2)}.$$

Knowing c and e, we may construct h and triangle ABC is easily obtained.

Also solved by *Walter B. Clarke*.

No. 362. Proposed by *N. A. Court*, University of Oklahoma.

Construct a tetrahedron so that its vertices shall lie on four given concurrent lines, and the respectively opposite faces shall meet the corresponding lines in four non-coplanar preassigned points.

Solution by the *Proposer*.

Let MAP, MBQ, MCR, MDS be the four given lines; P, Q, R, S the four preassigned points; and A, B, C, D the vertices of the required tetrahedron. The two tetrahedrons $ABCD$, $PQRS$ are perspective from the point M and from a plane and since $PQRS$ is inscribed in $ABCD$, their plane of perspectivity is the harmonic plane of M for both tetrahedrons.* Consequently, the faces of the required tetrahedron $ABCD$ are the planes determined by the vertices of the tetrahedron $PQRS$ and the lines of intersection of the respectively opposite faces of $PQRS$ with the harmonic plane of M for $PQRS$.

The problem may be stated as follows: construct the anticevian tetrahedron of a given point M for a given tetrahedron $PQRS$. A solution of the problem thus stated, less simple than the above solution, may be found in the American Mathematical Monthly, Vol. 43 (1936), p. 90.

Also solved by *Paul D. Thomas*.

PROPOSALS

No. 374. Proposed by *Paul D. Thomas*, Norman, Oklahoma.

Let Q be the foot of the perpendicular from the point P upon the polar of P with respect to the conic $Ax^2+By^2=C$.

(1) If P describes a straight line, then Q describes in general a cubic curve.

(2) If P describes a diameter of the conic, then Q describes an equilateral hyperbola.

(3) If P describes a line parallel to an axis of the given conic, then Q traces a circle with center on the other axis.

*See the Proposer's *Modern Pure Solid Geometry*, pp. 234-235.

No. 375. Proposed by *H. S. Grant*, Rutgers University.

What are the necessary and sufficient conditions that the polynomial

$$y=\sum_{i=0}^{n} a_i x^i, \quad a_i \text{ real}, \quad a_n \neq 0, \quad n \geqq 3,$$

be reducible to $Y=a_n X^n$ by a translation of axes. Where is the new origin?

No. 376. Proposed by *Walter B. Clarke*, San Jose, California.

The incircle of triangle ABC touches its sides at A_1, B_1, C_1. A_1B_1 is cut by the bisector of angle A at C_a, by the bisector of angle B at C_b; B_1C_1 is cut by the bisector of angle B at A_b, by the bisector of angle C at A_c; C_1A_1 is cut by the bisector of angle C at B_c, by the bisector of angle A at B_a. Show that A_bB_a, A_cC_a, and B_cC_b form the medial triangle of ABC.

No. 377. Proposed by *V. Thébault*, Le Mans, France.

Form two perfect squares whose sum shall be 148392.

No. 378. Proposed by *Robert C. Yates*, Louisiana State University.

An ellipse moves so that it is always tangent to two perpendicular lines. Find the locus of a focus.

No. 379. Proposed by *E. C. Kennedy*, Texas College of Arts and Industries.

Consider

$$T_n = \sqrt{\frac{k^2+T_{n-1}}{4-T_{n-1}}}, \quad 0<T_0<k/2.$$

What is the largest value of k^2 such the sequence $\{T_n\}$ converges to a real positive number? What is the number?

No. 380. Proposed by *N. A. Court*, University of Oklahoma.

The polar lines of a fixed line with respect to the spheres of a coaxal pencil lie on a quadric surface.

Bibliography and Reviews

Edited by
H. A. SIMMONS

Advanced Algebra. By S. Barnard and J. M. Child. Macmillan and Co., Lim., London, 1939. x+280 pages. $4.00.

This book is written as a continuation of *Higher Algebra* by the same authors. It treats many topics of advanced algebra and other branches of advanced mathematics, which are closely associated with algebra. Still other topics, in particular some purely algebraic ones, are supposed to be treated in special texts. The proper theory of *algebraic equations* is exemplified by a geometrical proof of the fundamental theorem of algebra (Chapter V). Furthermore, the theory of elimination is developed, after a short sketch of the general method, for some simple and instructive examples (Chapters II, VII). In a similar way, the formation of invariants and covariants is presented for first characteristic, important cases; and some fundamental properties of these functions are discussed (Chapter II, XVI). More completely presented is Gauss's solution of the equation $x^p-1=0$ for a prime number p (Chapter XIII). Besides the theory of primitive roots, which is necessary for that solution (Chapter XII), the authors develop some other subjects of number theory: quadratic residues, including Gauss's Lemma and the Law of Quadratic Reciprocity (Chapter X), the solution of quadratic Diophantine equations (Chapter XI), and the expression of integers as sums of squares (Chapter XIV). In connection with these developments, methods of factoring large numbers are discussed. Moreover the reader will find some chapters about advanced *analysis*. Chapter III treats double series with applications to power series and leads to the series of the elementary functions and to Bernouilli's numbers. Chapter IV discusses the notion of uniform convergence, Abel's Theorem for real and complex variables, applications to important series, and also Euler's constant. Considerations about conformal representation by elementary functions follow (Chapters V, VI). Continued fractions are treated in greater detail. In Chapter IX, the expression of a quadratic surd as a continued fraction is discussed. Chapter XV treats some types of continued fractions, develops convergence tests and, particularly, the transformation of continued fractions into equivalent forms. It treats the transformations of series into continued fractions, especially those of Euler and Lambert, and it gives the continued fractions for the most important functions. Even topics of *geometry* are treated, namely applications of linear substitutions in one variable (as cross ratios, homographic ranges, pencils, etc., Chapter I) and in two variables (some fundamental facts about projection and plane perspective, Chapter XVII). As another application, one finds a short chapter on probability (Chapter VIII).

This variety of topics is carefully explicated. The student of mathematics can learn a great deal by working through the book. The formulations and proofs are mostly precise and correct. A large number of examples at the close of the separate chapters and of the book contain interesting and well chosen problems. They will be extremely useful for the student.

Northwestern University. ERNST D. HELLINGER.

Bibliography of Mathematical Works Printed in America Through 1850. By L. C. Karpinski. The University of Michigan Press, Ann Arbor, Michigan. xxvi+697 pages. Price $6.00.

Although one may well be impressed by the magnitude of the present day production of mathematical literature in America, textbooks as well as scientific papers and treatises, it is sometimes hard to realize that this is but a continuation of a production that began more than two centuries ago and which had reached substantial proportions by 1850. The present moumental treatise gives an impressive account of this development.

Professor Karpinski describes his task as follows: "The bibliography includes not only the textbooks of arithmetic and algebra, but also those of geometry, trigonometry, analytical geometry, calculus, and, in general, of the college mathematics. All American publications of European works are given; they constitute an imposing mass of material. The list incorporates also the mathematical journals of which there were a half dozen in America before 1850."

The extent of the contribution is found in the fact that the volume lists 1092 separate publications and 1906 subsequent editions, a total of 2998 volumes. "One may reasonably conclude," says the author, "that the total number of books and pamphlets on mathematics printed in the Americas through 1850 is approximately 1,200... It may be assumed that from five to ten per cent have vanished entirely." In addition to books in Spanish, Portuguese, and Latin, one is surprised to find several in Hawaiian and an arithmetic in Choctaw.

The long history of mathematical publication in America is strikingly illustrated by the fact that the first work in the New World was published in 1556. This was a Mexican publication by Juan Diez Freyle entitled (in translation): "A brief summary of the reckoning of silver and gold.... With some rules relating to arithmetic." The book "includes algebraic problems, some pure number theory, and extensive arithmetical computations."

The first work in English appears to have been a superficial publication in 1703 by John Hill entitled, "The young secretary's guide: or, a speedy help to learning...". The twenty-fourth edition of this work appeared in 1750. The first English textbook on arithmetic to be printed in the New World was the reprint in 1719 of Hodder's arithmetic. According to the author "the two most widely used arithmetics of the eighteenth-century American publication are reprints of George Fisher's *The Instructor: or, Young Man's Best Companion*, and of Thomas Dilworth's *The Schoolmasters' Assistant.*" The comprehensive character of instruction in that early day is illustrated by the fact that the former volume also included "The Poor Planters' Physician...how to Pickle and Preserve; to make divers Sorts of Wine; and many excellent Plaisters."

Among entries of special interest one notes the numerous editions of Legendre's *Éléments de Géométrie*, which exerted so profound an influence upon the sequence of theorems in later American geometries. The first translation was made in 1819 by John Farrar. Among literary curiosities one finds that the second translation of Legendre, published in Edinburgh in 1824 and in New York in 1828, was made by Thomas Carlyle, although it was issued under the name of David Brewster.*

Perhaps the most interesting item in the bibliography refers to the translation and commentary of Laplace's *Mécanique Céleste* published by Nathaniel Bowditch.

*D. E. Smith and J. Ginsburg in their monograph: *A History of Mathematics in America Before 1900*, The Carus Mathematical Monographs, No. 5, 1934, give the date as 1822 for the Carlyle-Brewster translation of Legendre. The bibliography of this book furnishes many interesting commentaries.

This great work, containing about 4,000 pages, was the most monumental contribution to American mathematics in the perlord prior to 1850, and, the author might have added, it compares with some of the best work that has been done in the subsequent period.

The bibliography is beautifully printed and it contains as a special feature 908 zinc etchings reproducing practically all of the major titles. Referring to these engravings the author states that "bibliographers may regard this work as introducing a new method in their field."

Northwestern University. H. T. DAVIS.

A Brief Course in Trigonometry. By D. R. Curtiss and E. J. Moulton. D. C. Heath and Company, 1940. viii+118+17 pages. $1.50.

As the authors state in the preface, this book "is far from being a mere revision of our *Plane Trigonometry* published in 1927." The text is almost entirely rewritten and is very much shortened. The problems are more numerous and seem to be entirely new. It is designed for a two-hour semester course or the equivalent. As a result, some material usually found in trigonometries is omitted. The emphasis is on the theoretical, rather than on the triangle-solving, side of the subject. Logarithms and the solution of triangles by logarithms are given in the last two of the seven chapters. The solution of right triangles without the use of logarithms is given in Chapter II, which also includes a short paragraph on the non-logarithmic solution of certain oblique triangles. Answers, not always correct, are given for most of the odd-numbered problems. Figures are borrowed freely from the older book.

The general definitions of the trigonometric functions are introduced first, and their specialization for right triangles is made later. A list of the chapter headings will probably give an idea of the scope of the text. Chapter I, *Trigonometric Functions;* Chapter II, *Right Triangles;* Chapter III, *Reduction Formulas; Line Values, Graphs;* Chapter IV, *Trigonometric Identities* (274 problems); Cnapter V, *Radian Measure, Inverse Functions, Trigonometric Equations;* Chapter VI, *Logarithms, Four-Place Tables;* Chapter VII, *Solution of Triangles, Four-Place Logarithms.*

There are a number of misprints and errors, some of which are as follows. The degree mark (°) is too frequently omitted. The fourth drawing of Figure 13 has an error which instructors are not apt to notice, but which may cause the better students trouble. The worked example on pages 18-19 uses trigonometric tables to two significant figures, but gives the answer to three significant figures. This is not in accord with the usual practice, and contradicts a later section on computation. A rather peculiar rule is given for the "rounding-off" of numbers when the "dropped" part is exactly one-half. It is not one of the usual rules. The answer given for problem 7, page 24, cannot be obtained by use of the rule in the text. There is a parenthetical remark near the top of page 43 which is better omitted, since there is no angle θ for which $\sec\theta=0$. Figures 47 and 48 are poorly drawn (the graph of $y=\sec x$ seems to consist of arcs of circles!). Figure 61 (one of those for the "ambiguous" case) is poor and misleading since the two sides marked a are not equal to one another. Problem 1, page 102 is misprinted and should have $b=132.6$.

The four-place tables following the text seem to be suitable for the purpose. A useful and convenient table of squares (and, consequently, of square roots) is included. The table of natural functions includes the values of the secants and cosecants. The tables seem to have an "open" appearance which is pleasing to the eye. This is obtained by the omission of the first figure of each entry, except for certain

"leading" entries. To the reviewer, this is most unfortunate, as the tables are thus made more difficult to read. To a person used to this arrangement in the five (and higher) -place tables, it may not seem very bad, but it is very confusing to a beginner. Four-place tables do not need that device to save space. The inaccurate "proportional parts" columns in the table of logarithms is nowhere explained. They could well be omitted and the space utilized to fill in the first figures of the table entries. The "angle" columns for angles greater than 45° in the trigonometric tables are confusing to read, at least for a beginner. Table IV contains a reference to "Tables Va or Vb", which, unfortunately, are not included. Later printings and editons will, no doubt, have the misprints and errors corrected.

On the whole, the book seems to be well adapted to its purpose. Teachers with classes in trigonometry "meeting twice a week for a semester, or three or four times a week for a college quarter" should find the book suited to their purpose. The typography is good, the treatment is fine, and excellent use is made of italics and black type.

University of Arkansas. EDWIN COMFORT.

Elementary Theory of Equations. By William Vernon Lovitt, Prentice-Hall, Inc., New York, 1939. xi+237 pages. $2.50.

The text is intended for the use of students who have had no mathematical training beyond a one-semester course in analytic geometry, and is therefore written on a much more elementary level than are most books in this field. At the same time the material included is essentially the same as that in the standard texts, and, indeed, it contains some material, such as Græffe's method, which is not ordinarily included.

The book has much to commend it as a text for any undergraduate course in the subject, whether or not the students have had calculus. First of all, the proofs and discussion appear to be quite clear and comprehensible (although textbooks, like puddings, can be proved in one way only). Second, the problems are well chosen, and in sufficient number—an important criterion for any class text. And third, this reviewer is delighted to see Græffe's method included in an elementary textbook.

Unfortunately, the book is marred by certain, more or less superficial, defects, and it is regrettable to find these defects, which might have been so easily eradicated by a more careful check of the original manuscript, in a textbook that without them would deserve whole-hearted recommendation. To give examples, the phrase "graphical solution" of an equation is used in the sense of a ruler and compass construction of the roots (p. 200) and is also used where the construction does (for the quadratic on p. 24) and even must (for the cubic on p. 100) employ other tools. It would more nearly suit this reviewer's tastes if the expression "graphical solution" were reserved for the latter cases, and if "construction of the roots" were employed in the former. On p. 25 a certain two lines are said, without proof, to be tangent to a given parabola at such and such points. The omission of at least an indication of this proof is not in keeping with the style of the book as a whole. On p. 185, why and how are the "above formulas" to be applied to the "positive roots of the transformed equations"? They are, in fact, to be applied to *the* roots of this equation.

Somewhat more serious are the lapses in the treatment of a theorem due to Glenn James on limits of the roots of an equation. James himself states the theorem incorrectly by a confusion of notation which is easily detectable by the experienced reader. Lovitt partially corrects this, but leaves a misstatement that is somewhat more obscure. It is true that the illustrative examples are worked according to the theorem as it should be stated. If the student is not confused by this he certainly must become

so when in the proof, at the bottom of p. 114, the last inequality is said to be implied by the preceding one, whereas, the true implication is the reverse of this.

Nevertheless, such defects are not serious in a class text, since a watchful instructor can spot them in advance and point them out to his students. To this reviewer the merits indicated would be decisive. But the defects do impair the value of the text to anyone who might wish to learn the subject on his own.

University of Chicago. ALSTON S. HOUSEHOLDER.

Elementary Mathematics from an Advanced Standpoint, Geometry. By Felix Klein. Translated from the third German edition by E. R. Hedrick and C. A. Noble. The Macmillan Company, New York, 1939. ix+214 pages.

The first edition of Klein's *Elementarmathematik vom höheren Standpunkte aus* was published in 1908-09. Two other German editions followed, the last being published in 1925. Many reviews of this text have been written; one of the best is that of J. W. Young in the Bulletin of the American Mathematical Society, vol. 16 (1909-10), pages 254-265. It is therefore unnecessary here to give a complete account of the present translation of the second part of these lectures.

In 1932 E. R. Hedrick and C. A. Noble published a translation into English of the first volume of Klein's lectures, on arithmetic, algebra, and analysis. In the present text they cover the second volume of Klein's work with the omission of the final chapter on instruction in geometry, and the two appendices. The authors of this translation do not state whether they intend to bring out volume 3 also.

Klein's lectures were given to teachers of mathematics in secondary schools. In the first volume he had much to say about instruction in arithmetic, algebra, and analysis. In the second volume, what he had to say about instruction in geometry was largely relegated to the final part, which the translators have omitted. Volume 2 professes to give a survey of the entire field of geometry, so that teachers might see the subject as a whole. Coordinates are used almost everywhere.

Geometry has made great advances in the last thirty years. Can this work still claim to be "a survey of the entire field of geometry"? Is it still valuable as a reference text for teachers of mathematics? Should it be used in undergraduate "honor courses" or courses of independent reading?

Certainly, the entire field of modern geometry is not covered. A topologist would surely not be satisfied. Differential geometry occupies little space. The treatment of tensors, for example, leaves much to be desired. Nevertheless, the reader finds here an orientation in geometry which it would be hard to obtain elsewhere. The point of view of the chapters on foundations of geometry has been criticized (see the reviews of J. W. Young, referred to above).

This book is on a level above that attained by most of our teachers of secondary mathematics, but not above that which should have been reached by teachers in institutions of collegiate rank. Those who have the necessary background will find its reading well worth while. Only portions of it are available to an ordinary undergraduate student; extraordinary students will be able to master nearly all of it, and should find it stimulating.

The translation is excellently done, into idiomatic English. The reviewer has found the few errors too unimportant to be worth listing.

Northwestern University. D. R. CURTISS.

Theory of Probability. By Harold Jeffreys. At the Clarenden Press, Oxford, 1939. VI+380 pages.

In the beginning, the author gives rules which a consistent theory of induction must follow and presents axioms, conventions, and fundamental theorems.

Chapter II treats of direct probabilities through discussions of sampling, the Poisson Law, the normal law, Pearson Types, characteristic functions, the z-, t-, and and χ^2-distributions.

This is followed by a chapter on estimation problems. The author arrives at the solutions by finding certain posterior probablities,....when he is given certain information.

Chapter IV opens with the meaning of maximum likelihood, presents an approximation to maximum likelihood when sufficient statistics do not exist, and shows how to solve normal equations by the approximation method. The author explains the use of expectation, the meaning of Sheppard's correction, discusses the correction for the linear correlation coefficient, and touches lightly on randomization through randomized layout and Latin squares. The best part of the chapter is the discussion of rank correlation.

Chapter V presents significance tests. Here significance tests are made for various situations by finding the odds in favor of a certain value of a parameter. An elaborate discussion of these ideas is continued throughout Chapter VI. The approach to significance is quite different from that given by many authors.

Chapter VII contains interesting discussions of the present definitions of probability and points out the difference between estimation and significance problems.

Following a chapter on general questions concerning theories underlying probability is a set of Tables of K, which is used in significance tests; the value of K is the ratio of the probability of a certain hypothesis q to the probability of the hypothesis *not* q under certain conditions. These tables contain relations between the "number of observations" and the square of a variable for certain K-values (5 percent point, 1 percent point, etc.), which are used for arriving at significance tests. These tables are constructed so as to cover many situations.

The last chapter, in which a proof of Stirling's formula appears, treats of factorial functions.

The book will make a real contribution to the library of any mathematician.

Michigan State College W. D. BATEN.

Business Arithmetic for College Students. By William Schlauch. F. S. Crofts & Company, New York, 1939. VII+299 pages; $2.80.

Business Arithmetic for College Students is intended to give the commerce student a grounding in those aspects of mathematics applicable to business transactions. It is not stated that as a prerequisite for successful study of the book, the student should have had either college algebra or courses in accounting. The reviewer presumes that the course is intended for first year commerce students.

The book has evident merit but is not altogether free from defects. The obvious merits are:

A wide range of useful and well developed topic material. The arithmetic processes, business averages, percentage analysis, simple and compound interest, commercial discounts, depreciation methods, brokerage, foreign exchange, and financing plans are among the subjects covered. Particularly attractive are the altogether practical

chapters on fractions, measurements, business securities, financing methods, and statistical averages. On the whole, the author's choice of material to be presented has been good; the treatment of most of the topics is fairly exhaustive; and the balance between theoretical development and practical application is well maintained.

An adequate supply of homework exercises. A hundred and twenty-five sets of exercises, each containing from 3 to 15 problems, assure no lack of homework; and class discussion problems illustrate every point treated in the course.

Noticeable defects of the text are:

Too much algebra in both the development and summation of points of theory. It is the reviewer's experience that few commerce students have the pure mathematics background necessary for following algebraic treatment of subject matter. Those students without a previous course in college algebra might find the algebraic orientation of the text an insuperable difficulty.

Inclusion of some material which first year commerce students are not prepared to handle successfully. Thus Section 14 presumes the student to understand bookkeeping while the profit and loss analysis of pages 98-118 would be comprehensible only to a student with a strong background of accounting courses.

Failure to develop the compound interest topic of annuities. (Problems involving application of annuities). Tables of annuity values are given but the text contains nothing on the nature of or the mathematical development of annuities.

On the whole the text represents a welcome addition to the literature of college grade business mathematics. For commerce students who have had college algebra and say a year of college accounting the text has special merit.

Northwestern University. S. W. SPECTHRIE.

NAVAL ACADEMY GIVES CHAUVENET MEMORIAL

William Chauvenet, second chancellor of the University, is now commemorated by a plaque on the wall of Ridgley Arcade, presented by the United States Naval Academy and unveiled on Navy Day, October 27.

The plaque is a replica of one in Mahan Hall at the Academy, where Chauvenet is venerated as one of the founders of that institution. He left it to become professor of mathematics and astronomy here in 1852; was made chancellor in 1862 and served until his death in 1869.

Before an audience of some twelve hundred people in the Quadrangle, Capt. Mark C. Bowman, USN, presented the plaque on behalf of the Academy. It was unveiled by Louis Chauvenet, grandson of the former chancellor, and accepted by Chancellor George R. Throop. Capt. Bowman holds the chair of Seamanship and Navigation once occupied by Professor Chauvenet at the Academy.

The ceremonies constituted the local observance of Navy Day, and were attended by the St. Louis alumni of the Naval Academy, who were instrumental in arranging for the gift. Governor Lloyd C. Stark delivered the principal address, and the British Consul, Mr. H. B. McClelland, read a greeting from the Admiralty.—From Alumni Bulletin of the Washington University, October issue, 1940.